

Stability conditions and crepant small resolutions

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Abstract

In this paper, we describe the spaces of stability conditions on the triangulated categories associated to three dimensional crepant small resolutions. The resulting spaces have chamber structures such that each chamber corresponds to a birational model together with a special Fourier-Mukai transform. We observe that these spaces are covering spaces over certain open subsets of finite dimensional vector spaces, and determine their deck transformations.

1 Introduction

For a triangulated category \mathcal{D} , the notion of stability conditions on \mathcal{D} was introduced by T.Bridgeland [5] to give a mathematical framework for Douglas's notion of II-stability [13], [14]. Roughly speaking, Bridgeland's stability condition (Z, \mathcal{P}) on \mathcal{D} consists of a group homomorphism Z and full subcategories,

$$Z: K(\mathcal{D}) \longrightarrow \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D} \quad (\phi \in \mathbb{R}),$$

where $K(\mathcal{D})$ is the Grothendieck group of \mathcal{D} , and this pair satisfies some axioms. Given data as above, the objects of the subcategory $\mathcal{P}(\phi)$ are called semistable of phase ϕ , and this gives a generalization of the classical notion of semistable sheaves with a fixed slope on a smooth projective curve. In [5], T.Bridgeland showed that the set of locally finite stability conditions $\text{Stab}(\mathcal{D})$ has a natural topology, and proposed to study this space as a new categorical invariant. After he wrote the paper [5], some examples have been computed in [7], [8], [10], [29], [27], [26], [18], [2]. The purpose of this paper is to describe the spaces of stability conditions on some triangulated categories associated to three dimensional crepant small resolutions.

1.1 Crepant small resolutions

Let $Y = \text{Spec } R$ for a noetherian local complete Gorenstein \mathbb{C} -algebra R of dimension three, and $0 \in Y$ be the closed point. Assume that Y admits a resolution of singularities

$$f: X \rightarrow Y = \text{Spec } R$$

which is an isomorphism in codimension one. Then $f: X \rightarrow Y$ is called a *crepant small resolution*. It is well-known that $\omega_X = f^*\omega_Y$ and the exceptional locus $C = \cup_{i=1}^N C_i$ is a tree of rational curves. Let $D^b(X)$ be the bounded derived category of coherent sheaves on X . We define the triangulated subcategory $\mathcal{D}_{X/Y} \subset D^b(X)$ as follows:

$$\mathcal{D}_{X/Y} := \{E \in D^b(X) \mid \text{Supp}(E) \subset C\}.$$

The purpose of this paper is to study the space of stability conditions on $\mathcal{D}_{X/Y}$. Define the space $\text{Stab}(X/Y)$ by

$$\text{Stab}(X/Y) := \text{Stab}(\mathcal{D}_{X/Y}).$$

The simplest example is given by the morphism,

$$f: X \longrightarrow Y = \operatorname{Spec} \mathbb{C}[[x, y, z, w]]/(xy - zw)$$

which is the blowing up by the ideal $(x, z) \subset \mathcal{O}_Y$. In this case, the exceptional locus is a single rational curve, whose normal bundle is $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Taking the blow-up by the ideal $(x, w) \subset \mathcal{O}_Y$ gives another resolution $g: W \rightarrow Y$, and the diagram

$$X \xrightarrow{f} Y \xleftarrow{g} W,$$

is an example of a *flop* [25, Definition 6.10]. A new feature in studying $\operatorname{Stab}(X/Y)$ is that one has to take account of all the crepant small resolutions of Y , such as flops. We would like to explain this in two different contexts, string theory and minimal model theory.

1.2 Viewpoint from string theory

Here we give a rough picture on the space $\operatorname{Stab}(X/Y)$, from the viewpoint of string theory. We use notions of string theory, for example stringy Kähler moduli spaces, and SCFT. These notions are explained in [4, Section 2]. For instance assume X is a projective Calabi-Yau 3-fold, $\mathcal{D} = D^b(X)$, and $\mathcal{M}_K(X)$ is the stringy Kähler moduli space [4, (2.2)]. The space $\mathcal{M}_K(X)$ is a subspace of the moduli space of SCFT, and the associated topological B-twists are unchanged along $\mathcal{M}_K(X)$. Moreover if \hat{X} is a mirror of X , $\mathcal{M}_K(X)$ is supposed to be isomorphic to the moduli space of complex structures on \hat{X} . The space of stability conditions was introduced in order to understand $\mathcal{M}_K(X)$ mathematically. More precisely, it is believed that the quotient space of $\operatorname{Stab}(\mathcal{D})$ by the actions of $\operatorname{Aut}(\mathcal{D})$ and \mathbb{C} contains $\mathcal{M}_K(X)$ [4, Remark 3.9].

The moduli space of complexified Kähler forms $\beta + i\omega \in H^2(X, \mathbb{C})$ forms an open subset $\mathcal{U}_X \subset \mathcal{M}_K(X)$, which physicists call the *neighborhood of the large volume limit*. The important point is that there might be another topologically distinct Calabi-Yau 3-fold W such that \mathcal{U}_W is contained in $\mathcal{M}_K(X)$ in the moduli space of SCFT. In this case, the associated B-twists from the data of X and W are equivalent, therefore their categories of D -branes are also equivalent. Mathematically this means that there exists an equivalence of the bounded derived categories of coherent sheaves,

$$\Phi: D^b(W) \longrightarrow D^b(X).$$

In the language of algebraic geometry, W is called a *Fourier-Mukai partner* of X and a flop gives one example [3],[6],[11],[20]. Therefore in describing $\mathcal{M}_K(X)$, we have to take into account the neighborhoods of the large volume limits corresponding to several Fourier-Mukai partners.

In fact in the case of a flop $X \rightarrow Y \leftarrow W$, P. Aspinwall [1, Figure 2] describes the localized picture of $\mathcal{M}_K(X)$ assuming all the curves in X except the flopping curve $C \subset X$ are quite “big”. The resulting picture is a 2-sphere minus three points, $\mathcal{U}_X, \mathcal{U}_W$ are disjoint and their union is dense in the sphere. The string theory has singularities at one of the deleted points, and the other two points are large volume limits corresponding to X and W . Thus one should have a similar picture in the context of Bridgeland’s stability conditions. It seems that our localized category $\mathcal{D}_{X/Y}$ and the space $\operatorname{Stab}(X/Y)$ provide the right framework, thus we expect $\operatorname{Stab}(X/Y)$ is described via Fourier-Mukai partners such as flops.

1.3 Viewpoint from minimal model theory

Our interest also comes from the birational geometry, especially minimal model program, simply MMP [25], [22]. The MMP is a program aimed to find a good birational model for a given

projective variety, by contracting extraneous rational curves. The output is either a minimal model or Mori fiber space.

One of the points where the three dimensional MMP differs from the two dimensional one is that birational minimal models are not necessary unique, but connected by a sequence of flops [25, Theorem 6.38]. The philosophy of Y.Kawamata [19] is that one can capture the set of birational minimal models via a chamber structures on the *movable cone* [19, Definition 1.1]. According to [19, Theorem 2.3], chambers on the movable cone are given by the ample cones of birational minimal models.

Now let us consider two birational three dimensional minimal models $W \dashrightarrow X$. Then there exists an equivalence of bounded derived categories of coherent sheaves [3],[6],[11],[20],

$$\Phi: D^b(W) \longrightarrow D^b(X).$$

Let $\text{Stab}(X)$ be the space of stability conditions on $D^b(X)$. As a substitute for the ample cone, one expects to find a certain region $U_X \subset \text{Stab}(X)$, which corresponds to the neighborhood of the large volume limit in string theory. Then one can transfer stability conditions in $U_W \subset \text{Stab}(W)$ by the equivalence Φ to get the region $U(W, \Phi) \subset \text{Stab}(X)$. In summary for a birational minimal model W and an equivalence Φ as above, one obtains the correspondence,

$$(W, \Phi) \longmapsto U(W, \Phi) \subset \text{Stab}(X).$$

This picture is quite similar to the picture of the movable cone [19, Theorem 2.3]. Thus we guess the existence of the chamber structure on $\text{Stab}(X)$, which enables us to capture the pair (W, Φ) as above. Unfortunately, there are some technical issues in working with Bridgeland's stability conditions for the derived categories of projective 3-folds. In particular we do not know how to construct examples of stability conditions in this case. (Also see the last part of [4, Section 4].) Despite this problem, our category $\mathcal{D}_{X/Y}$ is quite amenable to studying stability conditions, and also sufficient for realizing our purpose.

1.4 The main results

Let $f: X \rightarrow Y$ be a crepant small resolution as in the first part of this introduction. For some technical reasons, we put the following additional assumption,

- There is a hyperplane section $0 \in Y_0 \subset Y$ such that its pull-back $X_0 := f^{-1}(Y_0)$ is smooth.

First we give the standard region in Definition 4.2 below,

$$U_X \subset \text{Stab}(X/Y),$$

which should correspond to the neighborhood of the large volume limit. Unfortunately, U_X is not an open subset of $\text{Stab}(X/Y)$, but open in the subspace of $\text{Stab}(X/Y)$ which we call the *normalized stability conditions*,

$$\text{Stab}_n(X/Y) := \{\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/Y) \mid Z([\mathcal{O}_x]) = -1\}.$$

In this paper, we give a description of the normalized version of the stability conditions. Because other stability conditions are obtained by the actions of the additive group \mathbb{C} from the normalized stability conditions, it is enough to study $\text{Stab}_n(X/Y)$ for our purpose. Here the action of \mathbb{C} is as follows: for $\lambda \in \mathbb{C}$ and $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/Y)$, $\lambda(\sigma) = (Z', \mathcal{P}')$ with $\mathcal{P}'(\phi) = \mathcal{P}(\phi + \text{Re } \lambda)$ and $Z'(E) = \exp(-i\pi\lambda)Z(E)$. Let $\text{Stab}_n^\circ(X/Y)$ be the connected component of $\text{Stab}_n(X/Y)$ which contains U_X .

Next we introduce the set $\text{FM}(X)$ to be the set of pairs (W, Φ) , where $g: W \rightarrow Y$ is a crepant small resolution, and

$$\Phi: D^b(W) \longrightarrow D^b(X)$$

gives an equivalence of derived categories, such that Φ is given in a special way, as described in Definition 3.13 below. By transferring the open set $U_W \subset \text{Stab}_n(W/Y)$ through the equivalence Φ , we give the following open set in Definition 4.7,

$$U(W, \Phi) \subset \text{Stab}_n(X/Y).$$

The following theorem realizes our purpose, that is the existence of the desired chamber structure on the space of stability conditions.

Theorem 1.1 *We have the following union of chambers:*

$$\mathcal{M} := \bigcup_{(W, \Phi) \in \text{FM}(X)} U(W, \Phi) \subset \text{Stab}_n^\circ(X/Y),$$

such that two chambers are either disjoint or equal. Moreover we have $\overline{\mathcal{M}} = \text{Stab}_n^\circ(X/Y)$.

By using the chamber structure given in Theorem 1.1, we show $\text{Stab}_n^\circ(X/Y)$ is a regular covering space over a certain open subset of a finite dimensional complex vector space. Let Λ_f be the root lattice associated to the exceptional locus of f . We denote by $N^1(X/Y)$ the group of numerical classes of \mathbb{R} -divisors and $N^1(X/Y)_\mathbb{C} := N^1(X/Y) \otimes_\mathbb{R} \mathbb{C}$. Then the elements of $N^1(X/Y)_\mathbb{C}$ are regarded as complex functions on Λ_f . Let $V(\Lambda_f)$ be the set of roots of Λ_f . For a root $v \in V(\Lambda_f)$ and $k \in \mathbb{Z}$, define H_v as follows:

$$H_v := \{\beta + i\omega \in N^1(X/Y)_\mathbb{C} \mid (\beta + i\omega)v \in \mathbb{Z}\}.$$

Theorem 1.2 *The map*

$$\mathcal{Z}_X: \text{Stab}_n^\circ(X/Y) \longrightarrow N^1(X/Y)_\mathbb{C} \setminus \bigcup_{v \in V(\Lambda_f)} H_v,$$

defined by sending a stability condition to its central charge, is a regular covering map.

The results of this paper together with some developments on derived categories and stability conditions are reviewed in T.Bridgeland's manuscript [9] for the ICM talk in 2006.

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2 Generalities

In this paper, all the schemes are defined over \mathbb{C} . For a scheme X , we denote by $\text{Coh}(X)$ and $D^b(X)$ the Abelian category of coherent sheaves and its bounded derived category respectively. The shift functor on $D^b(X)$ is denoted by $[1]$. Also for a subscheme $Z \subset X$, we denote by $\text{Coh}_Z(X) \subset \text{Coh}(X)$ the subcategory whose objects are supported on Z . For an object $E \in D^b(X)$, its support is defined by

$$\text{Supp}(E) := \bigcup_{p \in \mathbb{Z}} \text{Supp } H^p(E) \subset X.$$

For a triangulated category \mathcal{D} , its K -group is denoted by $K(\mathcal{D})$.

2.1 Stability conditions on triangulated categories

The notion of stability conditions on triangulated categories is introduced in [5] to give the mathematical framework for the Douglas's work on Π -stability [13], [14]. Here we collect the basic definitions and results in [5].

Definition 2.1 *A stability condition on a triangulated category \mathcal{D} consists of data $\sigma = (Z, \mathcal{P})$, where $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ is a linear map, and $\mathcal{P}(\phi) \subset \mathcal{D}$ is a full additive subcategory for each $\phi \in \mathbb{R}$, which satisfy the following:*

- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
- If $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$, then $\text{Hom}(A_1, A_2) = 0$.
- If $E \in \mathcal{P}(\phi)$ is non-zero, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.
- For a non-zero object $E \in \mathcal{D}$, we have the following collection of triangles:

$$\begin{array}{ccccccc} 0 = E_0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \xrightarrow{\quad} & \cdots \xrightarrow{\quad} E_n = E \\ & \nwarrow [1] & \swarrow & & \nwarrow [1] & \swarrow & & \nwarrow [1] & \swarrow \\ & & A_1 & & A_2 & & & & A_n \end{array}$$

such that $A_j \in \mathcal{P}(\phi_j)$ with $\phi_1 > \phi_2 > \cdots > \phi_n$.

Here Z is called the *central charge*. Each $\mathcal{P}(\phi)$ is an Abelian category, the non-zero objects of $\mathcal{P}(\phi)$ are called *semistable of phase ϕ* , and simple objects of $\mathcal{P}(\phi)$ are called *stable*. The objects A_j are called *semistable factors* of E with respect to σ . The following proposition is useful in constructing stability conditions.

Proposition 2.2 [5, Proposition 4.2] *Giving a stability condition on \mathcal{D} is equivalent to giving a heart of a bounded t -structure $\mathcal{A} \subset \mathcal{D}$, and a group homomorphism $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ called the *stability function*, such that for a non-zero object $E \in \mathcal{A}$ one has*

$$Z(E) \in \{r \exp(i\pi\phi) \mid r > 0, 0 < \phi \leq 1\},$$

and the pair (Z, \mathcal{A}) satisfies the Harder-Narasimhan property.

For the Harder-Narasimhan property, we refer [5, Definition 2.3]. Also a necessary condition for this property are found in [5, Proposition 2.4]. Given data (Z, \mathcal{A}) as above and a non-zero object $E \in \mathcal{A}$, we define $\phi(E) \in (0, 1]$ such that $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi(E))$ uniquely. We call $\phi(E)$ the *phase* of E .

Remark 2.3 The condition (b) of [5, Proposition 2.4] is satisfied if \mathcal{A} is noetherian, i.e. for $E \in \mathcal{A}$, there are no infinite sequences of subobjects in \mathcal{A} ,

$$E_1 \subset E_2 \subset \cdots E_j \subset \cdots \subset E.$$

Therefore in order to check the Harder-Narasimhan property, it is enough to check that \mathcal{A} is noetherian and the condition (a) of [5, Proposition 2.4], i.e. there are no infinite sequence of subobjects in \mathcal{A} ,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1,$$

with $\phi(E_{j+1}) > \phi(E_j)$ for all j .

The set of stability conditions which satisfies the technical condition *local finiteness* [5, Definition 5.7] is denoted by $\text{Stab}(\mathcal{D})$. It is shown in [5, Section 6] that $\text{Stab}(\mathcal{D})$ has a natural topology. Forgetting the information of \mathcal{P} , we have the map

$$\mathcal{Z}: \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}).$$

Theorem 2.4 [5, Theorem 1.2] *For each connected component $\Sigma \subset \text{Stab}(\mathcal{D})$, there exists a linear subspace $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$, such that \mathcal{Z} restricts to a local homeomorphism, $\mathcal{Z}: \Sigma \rightarrow V(\Sigma)$.*

In general $\text{Stab}(\mathcal{D})$ is infinite dimensional, so we usually consider only numerical stability conditions as in [5], [7]. But if we assume $K(\mathcal{D})$ is finitely generated, Theorem 2.4 implies each connected component $\Sigma \subset \text{Stab}(\mathcal{D})$ is a complex manifold.

3 Geometry on crepant small resolutions

Let $f: X \rightarrow Y = \text{Spec } R$ be a three dimensional crepant small resolution as in the introduction. The exceptional locus $C \subset X$ is a tree of rational curves,

$$C = C_1 \cup C_2 \cup \cdots \cup C_N,$$

with each C_i isomorphic to \mathbb{P}^1 . (See for example [12, Lemma 3.4.1].) In this paper, we put the following additional assumption.

- There exists a hyperplane section $0 \in Y_0 \subset Y$ such that its pull-back $X_0 := f^{-1}(Y_0)$ is smooth.

In general, it is known that a general hyper plane section Y_0 is a rational double point, and $X_0 \rightarrow Y_0$ is a partial resolution, i.e. the minimal resolution $X'_0 \rightarrow Y_0$ factors through $X'_0 \rightarrow X_0 \rightarrow Y_0$. (See [28, (1.1), (1.14)].) Also it is known that for a given crepant small resolution $f: X \rightarrow Y$, there exists a finite map $Y' \rightarrow Y$ such that Y' admits a crepant small resolution which satisfies the above assumption. (See [25, Theorem 4.28]). The above assumption will be required in Lemma 3.8, Subsection 5.1 and Lemma 6.3. Of course this assumption is satisfied

in the case of the resolution of the ordinary double point described in the introduction. As in the introduction, we define $\mathcal{D}_{X/Y}$ to be

$$\mathcal{D}_{X/Y} := \{E \in D^b(X) \mid \text{Supp } E \subset C\}.$$

We collect some notation and known results on this resolution, and gives some lemmas. All the lemmas in this section will be proved in Section 6.

3.1 Terminology and results from birational geometry

Here we introduce standard terminology in birational geometry, for example used in [19, Definition 1.1]. Two divisors D_1, D_2 on X are called *numerically equivalent* over Y if and only if $D_1 \cdot C_i = D_2 \cdot C_i$ for all $1 \leq i \leq N$. Similarly, two 1-cycles Z_1, Z_2 on X contracted by f are *numerically equivalent* if and only if $D \cdot Z_1 = D \cdot Z_2$ for every divisor D on X .

Definition 3.1 We define the \mathbb{R} -vector spaces $N^1(X/S), N_1(X/S)$ to be

$$\begin{aligned} N^1(X/S) &:= \{\text{Divisors on } X\} / (\text{numerical equivalence over } Y) \otimes_{\mathbb{Z}} \mathbb{R}, \\ N_1(X/S) &:= \{\text{One cycles on } X \text{ contracted by } f\} / (\text{numerical equivalence}) \otimes_{\mathbb{Z}} \mathbb{R} \end{aligned}$$

By the definition, one has the perfect pairing,

$$N^1(X/Y) \times N_1(X/S) \ni (D, Z) \longmapsto D \cdot Z \in \mathbb{R}.$$

Moreover since we are assuming Y is complete, there exist divisors D_i on X for $1 \leq i \leq N$ such that

$$D_i \cdot C_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

by [12, Lemma 3.4.4]. Therefore we have

$$N^1(X/S) = \bigoplus_{1 \leq i \leq N} \mathbb{R}[D_i], \quad N_1(X/S) = \bigoplus_{1 \leq i \leq N} \mathbb{R}[C_i],$$

in our case. Similarly we introduce the one dimensional \mathbb{R} -vector spaces,

$$N^0(X/Y) = \mathbb{R}[X], \quad N_0(X/Y) = \mathbb{R}[p],$$

for a closed point $p \in C$ as *numerical classes of codimension zero cycles* and *zero dimensional cycles*. We have the pairing,

$$N^0(X/Y) \times N_0(X/Y) \ni (a[X], b[p]) \longmapsto ab \in \mathbb{R}.$$

Let $N^1(X/Y)_{\mathbb{C}} := N^1(X/Y) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 3.2 We define the ample cone $A(X/Y)$ and the complexified ample cone $A(X/Y)_{\mathbb{C}}$ to be

$$\begin{aligned} A(X/Y) &:= \{\text{Numerical classes of ample } \mathbb{R}\text{-divisors}\} \subset N^1(X/Y) \\ &= \bigoplus_{1 \leq i \leq N} \mathbb{R}_{>0}[D_i], \end{aligned}$$

$$A(X/Y)_{\mathbb{C}} := \{\beta + i\omega \in N^1(X/Y)_{\mathbb{C}} \mid \omega \in A(X/Y)\}.$$

Also for each $1 \leq j \leq N$ and $k \in \mathbb{Z}$, define \mathcal{W}_j and $\mathcal{W}_{j,k}$ to be

$$\begin{aligned} \mathcal{W}_j &:= \{\beta + i\omega \in N^1(X/Y)_{\mathbb{C}} \mid \omega \cdot C_j = 0 \text{ and } \omega \cdot C_{j'} > 0 \text{ for } j \neq j'\} \\ \mathcal{W}_{j,k} &:= \{\beta + i\omega \in \mathcal{W}_j \mid \beta \cdot C_j \in (k-1, k)\}. \end{aligned}$$

Note that the union $\bigcup_{1 \leq j \leq N} \mathcal{W}_j$ is the codimension one boundary of the *complexified nef cone* $\overline{A}(X/Y)_{\mathbb{C}} \subset N^1(X/Y)_{\mathbb{C}}$. For each $1 \leq i \leq N$, some multiple of the divisor

$$\sum_{j \neq i} D_j,$$

is base point free by [25, Theorem 3.3], thus there exists a birational contraction $g_i: X \rightarrow Y_i$ which contracts only C_i . Then \mathcal{W}_i is written as

$$\mathcal{W}_i = g_i^* \mathcal{A}(Y_i/Y)_{\mathbb{C}}.$$

Furthermore one can construct its *flop* [25, Theorem 6.14] and obtain the diagram below:

$$\begin{array}{ccccc} (C_i \subset X) & \xrightarrow{g_i} & (p_i \in Y_i) & \xleftarrow{g_i^\dagger} & (X_i^\dagger \supset C_i^\dagger) \\ & \searrow f & \downarrow h_i & \swarrow f_i^\dagger & \\ & & Y & & \end{array} \quad (1)$$

Next let $g: W \rightarrow Y$ be another crepant small resolution, and $\phi = f^{-1} \circ g: W \dashrightarrow X$ be the birational map. Because ϕ is an isomorphism in codimension one, one has the isomorphism of the groups of divisor classes, $\phi_*: N^1(W/Y) \rightarrow N^1(X/Y)$ called the *strict transform*. We use the same notation $\phi_*: \text{Pic}(W) \rightarrow \text{Pic}(X)$ for the isomorphism of Picard groups. The following theorem gives a chamber structure on $N^1(X/Y)$, where each chamber corresponds to the ample cone of a crepant small resolution of Y .

Theorem 3.3 [23, Main theorem], [19, Theorem 2.3] *The number of crepant small resolutions $g: W \rightarrow Y$ is finite up to isomorphism. One has the decomposition,*

$$N^1(X/Y) = \bigcup_{(W, \phi)} \phi_* \overline{A}(W/Y).$$

Here (W, ϕ) is a pair of a crepant small resolution $g: W \rightarrow Y$ and the birational map $\phi = f^{-1} \circ g$. Moreover $\phi_* \mathcal{A}(W/Y) \cap \phi'_* \mathcal{A}(W'/Y) \neq \emptyset$ if and only if there exists an isomorphism $h: W' \rightarrow W$ such that $\phi \circ h = \phi'$.

Remark 3.4 In our case, $f: X \rightarrow Y$ is an isomorphism in codimension one. Therefore the f -effective f -movable cone and f -effective f -nef cone defined in [19, Definition 1.1] coincide with $N^1(X/Y)$ and $\overline{A}(X/Y)$ respectively.

3.2 Chern characters and Riemann-Roch theorem

We consider the chern characters which take values in our vector spaces $N^*(X/Y)$ and $N_*(X/Y)$,

$$\begin{aligned} \mathcal{D}^b(X) \ni E &\longmapsto (\text{ch}_0(E), \text{ch}_1(E)) \in N^0(X/Y) \oplus N^1(X/Y), \\ \mathcal{D}_{X/Y} \ni F &\longmapsto (\text{ch}_2(F), \text{ch}_3(F)) \in N_1(X/Y) \oplus N_0(X/Y). \end{aligned}$$

Usually, chern characters take values in Chow groups. But since rationally equivalent two cycles are numerically equivalent, ch_0, ch_1 can take values in $N^0(X/Y), N^1(X/Y)$. Also $F \in \mathcal{D}_{X/Y}$ is supported on C , thus ch_2, ch_3 take values in $N_1(X/Y), N_0(X/Y)$.

Let us take $E \in D^b(X)$, $F \in \mathcal{D}_{X/Y}$. Since the support of F is proper, the space $\text{Ext}_X^i(E, F)$ is finite dimensional, and zero except for a finite number of $i \in \mathbb{Z}$. Therefore the number

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}_X^i(E, F) \in \mathbb{Z},$$

makes sense. The Riemann-Roch theorem [15, Corollary 18.3.1] implies,

$$\chi(E, F) = \text{ch}_0(E) \cdot \text{ch}_3(F) - \text{ch}_1(E) \cdot \text{ch}_2(F).$$

3.3 Perverse t-structures

Here we introduce the Abelian categories ${}^p\text{Per}(X/Y) \subset D^b(X)$ for $p = 0, -1$, which are introduced in [6] to construct the flops. These categories provide a central technical tool in this paper. For the precise definition, we refer [6, Section 3] and [12, Section 3]. First we give the known results of [6],[12]. Let S_0 , S'_0 , and S_i for $1 \leq i \leq N$ be

$$S_0 := \omega_C[1], \quad S'_0 := \mathcal{O}_C, \quad S_i := \mathcal{O}_{C_i}(-1).$$

Here the scheme structure on C_i is reduced in the definition of S_i for $1 \leq i \leq N$. However the scheme structure on C in the definition of S_0 , S'_0 must be the scheme theoretic fiber of $f: X \rightarrow Y$, thus possibly non-reduced.

Lemma 3.5 [6, Lemma 3.2], [12, Section 3] *For $p = 0$ or -1 , there exist hearts of bounded t-structures ${}^p\text{Per}(X/Y) \subset D^b(X)$, such that $E \in D^b(X)$ belongs to ${}^p\text{Per}(X/Y)$ if and only if the following holds:*

- (i) E is concentrated in degrees $[-1, 0]$.
- (ii) $f_* H^{-1}(E) = 0$ and $R^1 f_* H^0(E) = 0$.
- (iii) For $j \geq 1$, we have the following:

$$\text{Hom}(S_j, H^{-1}(E)) = 0, \quad (p = 0), \quad \text{Hom}(H^0(E), S_j) = 0, \quad (p = -1).$$

Here we define the categories ${}^p\text{Per}(\mathcal{D}_{X/Y})$ for $p = -1, 0$ to be

$${}^p\text{Per}(\mathcal{D}_{X/Y}) := {}^p\text{Per}(X/Y) \cap \mathcal{D}_{X/Y}.$$

M. Van den Bergh [12] determined the simple objects of these categories.

Proposition 3.6 [12, Proposition 3.5.8] *The categories*

$${}^0\text{Per}(\mathcal{D}_{X/Y}), \quad \text{and} \quad {}^{-1}\text{Per}(\mathcal{D}_{X/Y})$$

are hearts of bounded t-structures on $\mathcal{D}_{X/Y}$, and are finite length Abelian categories. Their simple objects are

$$\{S_i\}_{0 \leq i \leq N}, \quad \text{and} \quad \{S'_0, S_i[1]\}_{1 \leq i \leq N}$$

respectively.

Let $g_i: X \rightarrow Y_i$ be the contraction which contracts C_i , as in the diagram (1). Note that one can define ${}^p\text{Per}(X/Y_i) \subset D^b(X)$, and ${}^p\text{Per}(\mathcal{D}_{X/Y_i}) \subset \mathcal{D}_{X/Y_i}$ similarly. We introduce the following subcategories of $\mathcal{D}_{X/Y}$.

Definition 3.7 For $1 \leq i \leq N$ and $k \in \mathbb{Z}$, we define $\mathcal{A}_{(i,k)}$ by

$$\mathcal{A}_{(i,k)} := ({}^0\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y}) \otimes \mathcal{O}_X(kD_i).$$

The following lemma will be used in the next section. We give the proof in Section 6.

Lemma 3.8 (i) For $p = -1, 0$, we have

$$\text{Coh}_{\cup_{j \neq i} C_j}(X) \cup {}^p\text{Per}(\mathcal{D}_{X/Y_i}) \subset {}^p\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y},$$

as subcategories of $\mathcal{D}_{X/Y}$. Moreover the right hand side coincides with the smallest extension closed subcategory of $\mathcal{D}_{X/Y}$ which contains the left hand side.

(ii) We have the following equality of subcategories of $\mathcal{D}_{X/Y}$:

$${}^{-1}\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y} = \mathcal{A}_{(i,1)}.$$

(iii) Any object in $\mathcal{A}_{(i,k)}$ is given by a successive extension of the objects in

$$\text{Coh}_{\cup_{j \neq i} C_j}(X), \quad \mathcal{O}_{C_i}(k-2)[1], \quad \mathcal{O}_{C_i}(k-1).$$

Here C_i has a reduced scheme structure.

Remark 3.9 As we see in Section 6, the assumption that X_0 is smooth is needed in the proof of Lemma 3.8 (ii), (iii). If otherwise C_i in Lemma 3.8 (iii) may not be reduced, so the arguments in the next section (especially Lemma 4.3) does not work.

3.4 Fourier-Mukai transforms and standard equivalences

The Fourier-Mukai transform is a useful tool in studying derived categories. Let $g: W \rightarrow Y$ be another crepant resolution. We introduce the relative version of the Fourier-Mukai transform.

Definition 3.10 We say the equivalence $\Phi: D^b(W) \rightarrow D^b(X)$ is of Fourier-Mukai type over Y if there exists an object $\mathcal{E} \in D^b(X \times W)$, which is supported on $X \times_Y W$, such that Φ is written as

$$\Phi \cong \Phi_{W \rightarrow X}^{\mathcal{E}} := \mathbf{R}p_{X*}(p_W^*(*) \stackrel{L}{\otimes} \mathcal{E}).$$

Here p_X, p_W are corresponding projections from $X \times W$. The object $\mathcal{E} \in D^b(X \times W)$ is called the kernel of Φ .

The following theorem plays an important role in this paper.

Theorem 3.11 [6, Theorem 1.1], [11, Proposition 4.2] Let $g: W \rightarrow Y$ be another crepant small resolution, and $\phi: W \dashrightarrow X$ be the birational map. Assume for a g -ample divisor H on W , the divisor $-\phi_*H$ is f -ample. Then the functor

$$\Phi_{W \rightarrow X}^{\mathcal{O}_{W \times_Y X}}: D^b(W) \longrightarrow D^b(X),$$

gives an equivalence, and takes ${}^{-1}\text{Per}(W/Y)$ to ${}^0\text{Per}(X/Y)$.

We call the equivalence given in Theorem 3.11 *standard*.

Remark 3.12 Let us apply Theorem 3.11 to the sequence $X_i^\dagger \rightarrow Y_i \leftarrow X$ as in the diagram (1). Then there exists a standard equivalence

$$D^b(X_i^\dagger) \longrightarrow D^b(X),$$

which takes $^{-1}\text{Per}(X_i^\dagger/Y_i)$ to $^0\text{Per}(X/Y_i)$ but not necessary takes $^{-1}\text{Per}(X_i^\dagger/Y)$ to $^0\text{Per}(X/Y)$.

We introduce the set $\text{FM}(X)$ as follows.

Definition 3.13 We define $\text{FM}(X)$ to be the set of pairs (W, Φ) , where $g: W \rightarrow Y$ is another crepant small resolution and $\Phi: D^b(W) \rightarrow D^b(X)$ is an equivalence of derived categories which satisfies the following: there exists a factorization of the birational map $\phi: W \dashrightarrow X$,

$$W = X^n \dashrightarrow X^{n-1} \dashrightarrow \cdots \dashrightarrow X^1 \dashrightarrow X^0 = X,$$

and equivalences of Fourier-Mukai type over Y , $\Phi^j: D^b(X^j) \rightarrow D^b(X^{j-1})$ such that $\Phi \cong \Phi^1 \circ \cdots \circ \Phi^n$. Each Φ^j is one of the following:

- type I : $X^j = X^{j-1}$ and $\Phi^j \cong \otimes \mathcal{L}$ for $\mathcal{L} \in \text{Pic}(X^j)$.
- type II : $X^j \dashrightarrow X^{j-1}$ is a flop at a single rational curve as in the diagram (1), and Φ^j is a standard equivalence.

Note that for $(W, \Phi) \in \text{FM}(X)$, Φ restricts to the equivalence,

$$\Phi: \mathcal{D}_{W/Y} \longrightarrow \mathcal{D}_{X/Y},$$

because the kernel of Φ is supported on $X \times_Y W$. Therefore Φ induces an isomorphism between $K(\mathcal{D}_{W/Y})$ and $K(\mathcal{D}_{X/Y})$. For the description of $K(\mathcal{D}_{X/Y})$, we have the following lemma. We give the proof in Section 6.

Lemma 3.14 (i) The class $[\mathcal{O}_x] \in K(\mathcal{D}_{X/Y})$ does not depend on a choice of $x \in C$, and $K(\mathcal{D}_{X/Y})$ is described by the direct sum,

$$K(\mathcal{D}_{X/Y}) = \mathbb{Z}[\mathcal{O}_x] \oplus \bigoplus_{i=1}^N \mathbb{Z}[\mathcal{O}_{C_i}(-1)].$$

(ii) For $(W, \Phi) \in \text{FM}(X)$, Φ takes the class $[\mathcal{O}_w] \in K(\mathcal{D}_{W/Y})$ to $[\mathcal{O}_x] \in K(\mathcal{D}_{X/Y})$ for closed points $w \in W$ and $x \in X$.

Finally we give the following lemma, which relates Fourier-Mukai transforms to chern characters. The proof will also be given in Section 6.

Lemma 3.15 For $(W, \Phi) \in \text{FM}(X)$ and $\mathcal{L} \in \text{Pic}(W)$, one has

$$\text{ch}_1 \Phi(\mathcal{L}) = \phi_* \text{ch}_1 \mathcal{L} + \text{ch}_1 \Phi(\mathcal{O}_W).$$

Here $\phi: W \dashrightarrow X$ is the birational map.

4 The chamber structures on the normalized stability conditions

In this section, we prove Theorem 1.1. As in the introduction, let $\text{Stab}(X/Y)$ be the space of stability conditions on $\mathcal{D}_{X/Y}$.

4.1 Constructions of stability conditions

Here we construct stability conditions, which corresponds to the neighborhood of the large volume limit. By Proposition 2.2, in order to give a stability condition, we find a heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}_{X/Y}$ and a stability function $Z: K(\mathcal{D}_{X/Y}) \rightarrow \mathbb{C}$. As a heart of a bounded t-structure, we take the subcategory

$$\text{Coh}_C(X) = \text{Coh}(X) \cap \mathcal{D}_{X/Y} \subset \mathcal{D}_{X/Y}.$$

In order to construct stability functions, let us take an element $\beta + i\omega \in N^1(X/Y)_{\mathbb{C}}$. We define $Z_{(\beta, \omega)}: K(\mathcal{D}_{X/Y}) \rightarrow \mathbb{C}$ as follows:

$$\begin{aligned} Z_{(\beta, \omega)}(E) &:= - \int e^{-(\beta + i\omega)} \text{ch}(E) \\ &= -\text{ch}_3(E) + (\beta + i\omega) \text{ch}_2(E). \end{aligned}$$

Lemma 4.1 *Assume $\beta + i\omega \in A(X/Y)_{\mathbb{C}}$. Then the pair*

$$\sigma_{(\beta, \omega)} := (Z_{(\beta, \omega)}, \text{Coh}_C(X))$$

determines a point of $\text{Stab}(X/Y)$.

Proof. We use Proposition 2.2 and Remark 2.3. Let us take a non-zero object $E \in \text{Coh}_C(X)$. If $\dim \text{Supp}(E) = 1$, then $\text{Im } Z_{(\beta, \omega)}(E) > 0$ since ω is ample. If $\dim \text{Supp}(E) = 0$, then $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{<0}$. Therefore it suffices to show the Harder-Narasimhan property.

By Remark 2.3, we check $\text{Coh}_C(X)$ is noetherian. But this is obvious, because a finitely generated module over a noetherian ring is noetherian. Thus it is enough to check that there are no infinite sequence,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1,$$

for $E_j \in \text{Coh}_C(X)$ with $\phi(E_{j+1}) > \phi(E_j)$ for all j . Assume the sequence as above exists. For $1 \leq i \leq N$, let $\eta_i \in X$ be the generic point of C_i . Then since E_j are supported on C , one has

$$0 \leq \text{length}_{\mathcal{O}_{X, \eta_i}} E_{j+1} \leq \text{length}_{\mathcal{O}_{X, \eta_i}} E_j < \infty.$$

Therefore we may assume $\text{length}_{\mathcal{O}_{X, \eta_i}} E_j$ is constant for all i and j . Then if we take the exact sequence,

$$0 \longrightarrow E_{j+1} \longrightarrow E_j \longrightarrow G_j \longrightarrow 0,$$

then G_j is zero dimensional, thus $Z_{(\beta, \omega)}(G_j) \in \mathbb{R}_{\leq 0}$. This implies $\phi(E_j) \geq \phi(E_{j+1})$, thus we get a contradiction. \square

The stability conditions $\sigma_{(\beta, \omega)}$ are contained in the following subspace:

$$\text{Stab}_n(X/Y) := \{\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/Y) \mid Z([\mathcal{O}_x]) = -1\},$$

for closed points $x \in C$. We call $\text{Stab}_n(X/Y)$ the set of *normalized stability conditions*. By Lemma 3.14 (i), the following map gives an isomorphism of the vector spaces:

$$N^1(X/Y)_{\mathbb{C}} \ni \beta + i\omega \longmapsto Z_{(\beta, \omega)} \in \{Z \in \text{Hom}(K(\mathcal{D}_{X/Y}), \mathbb{C}) \mid Z([\mathcal{O}_x]) = -1\}.$$

Hence the map sending stability conditions to their central charges restricts to give the map,

$$\mathcal{Z}_X: \text{Stab}_n(X/Y) \longrightarrow N^1(X/Y)_{\mathbb{C}}.$$

We give the following definition.

Definition 4.2 We define $U_X \subset \text{Stab}_n(X/Y)$ as follows,

$$U_X := \{\sigma_{(\beta, \omega)} \in \text{Stab}_n(X/Y) \mid \beta + i\omega \in A(X/Y)_{\mathbb{C}}\}.$$

Also define

$$\text{Stab}_n^{\circ}(X/Y) \subset \text{Stab}_n(X/Y),$$

to be the connected component of $\text{Stab}_n(X/Y)$ which contains U_X .

Note that \mathcal{Z}_X restricts to a homeomorphism between U_X and $A(X/Y)_{\mathbb{C}}$.

4.2 Codimension one boundaries of U_X

Here we give the descriptions of codimension one boundaries of the domain U_X . Let us take a point,

$$\sigma = (Z_{(\beta, \omega)}, \mathcal{A}) \in \overline{U}_X.$$

Note that by the continuity of \mathcal{Z}_X , we have

$$\beta + i\omega = \mathcal{Z}_X(\sigma) \in \overline{A}(X/Y)_{\mathbb{C}}.$$

We say σ lies in the codimension one boundary if $\mathcal{Z}_X(\sigma) \in \mathcal{W}_i$ for some $1 \leq i \leq N$. The purpose here is to find a heart of a t-structure \mathcal{A} such that the pair $(Z_{(\beta, \omega)}, \mathcal{A})$ gives a stability condition, and lies in the codimension one boundary. Recall that we defined $\mathcal{W}_{i,k}$ in Definition 3.2 and $\mathcal{A}_{(i,k)}$ in Definition 3.7.

Lemma 4.3 Let us take $\beta + i\omega \in \mathcal{W}_{i,k}$ for some $k \in \mathbb{Z}$. Then the pair

$$\sigma_{(\beta, \omega)} := (Z_{(\beta, \omega)}, \mathcal{A}_{(i,k)}),$$

gives a stability condition on $\mathcal{D}_{X/Y}$.

Proof. First we check the following,

$$Z_{(\beta, \omega)}(\mathcal{A}_{(i,k)}) \subset \{\mathbb{R}_{>0} \exp(i\pi\phi) \mid 0 < \phi \leq 1\}. \quad (2)$$

By Lemma 3.8 (iii), $\mathcal{A}_{(i,k)}$ is the smallest extension closed subcategory of $\mathcal{D}_{X/Y}$ which contains

$$\text{Coh}_{\cup_{j \neq i} C_j}(X), \quad \mathcal{O}_{C_i}(k-2)[1], \quad \mathcal{O}_{C_i}(k-1).$$

Hence it is enough to check (2) for the above objects. Let us take $E \in \text{Coh}_{\cup_{j \neq i} C_j}(X)$. If $\dim \text{Supp}(E) = 1$, then $\text{Im } Z(E) > 0$ because $\omega \cdot C_j > 0$ for $j \neq i$. If $\dim \text{Supp}(E) = 0$, one has $Z_{(\beta, \omega)}(E) \in \mathbb{R}_{<0}$. Therefore (2) holds for $E \in \text{Coh}_{\cup_{j \neq i} C_j}(X)$. For other two objects, one has

$$\begin{aligned} Z_{(\beta, \omega)}(\mathcal{O}_{C_i}(k-2)[1]) &= k-1-\beta \cdot C_i \in \mathbb{R}_{<0}, \\ Z_{(\beta, \omega)}(\mathcal{O}_{C_i}(k-1)) &= -k+\beta \cdot C_i \in \mathbb{R}_{<0}. \end{aligned}$$

Therefore (2) holds, and it is enough to check $\mathcal{A}_{(i,k)}$ is noetherian and the pair $(Z_{(\beta, \omega)}, \mathcal{A}_{(i,k)})$ satisfies Harder-Narasimhan property, by Remark 2.3.

First we check $\mathcal{A}_{(i,k)}$ is noetherian. Assume there exists an infinite sequence

$$E_1 \subset E_2 \subset \cdots E_n \subset \cdots \subset E$$

in $\mathcal{A}_{(i,k)}$. We put $E_\infty = E$. By Lemma 3.8 (iii), the complex E_n is concentrated in degree zero at the generic point of C_j for $j \neq i$. Hence we may assume the generic length of E_n at C_j are constant for all $j \neq i$. Then let us take the exact sequence

$$0 \rightarrow E_1 \rightarrow E_n \rightarrow G_n \rightarrow 0$$

in $\mathcal{A}_{(i,k)}$. Then G_n is supported on C_i , hence

$$G_n \in {}^0\text{Per}(\mathcal{D}_{X/Y_i}) \otimes \mathcal{O}_X(kD_i).$$

We obtain the sequence

$$G_1 \subset G_2 \subset \cdots \subset G_\infty,$$

in ${}^0\text{Per}(\mathcal{D}_{X/Y_i}) \otimes \mathcal{O}_X(kD_i)$. By Proposition 3.6 ${}^0\text{Per}(\mathcal{D}_{X/Y_i}) \otimes \mathcal{O}_X(kD_i)$ is finite length, hence this sequence terminates.

Finally we check the Harder-Narasimhan property. Let us assume there exists an infinite sequence

$$\cdots \subset E_n \subset \cdots \subset E_2 \subset E_1,$$

in $\mathcal{A}_{(i,k)}$ such that $\phi(E_{n+1}) > \phi(E_n)$ for all n . Then $\text{Im } Z_{(\beta,\omega)}(E_n) > 0$. Again we may assume generic length of E_n at C_j are constant for $j \neq i$. If we take the exact sequence

$$0 \rightarrow E_{n+1} \rightarrow E_n \rightarrow G_n \rightarrow 0$$

in $\mathcal{A}_{(i,k)}$, then G_n is supported on C_i , hence $Z_{(\beta,\omega)}(G_n) \in \mathbb{R}_{\leq 0}$. But this implies $\phi(E_n) \geq \phi(E_{n+1})$, hence a contradiction. \square

The following proposition gives the complete description of the codimension one boundaries of U_X .

Proposition 4.4 *For $\beta + i\omega \in \mathcal{W}_i$, there exists a point $\sigma \in \overline{U}_X$ with $\mathcal{Z}_X(\sigma) = \beta + i\omega$ if and only if $\beta + i\omega \in \mathcal{W}_{i,k}$ for some $k \in \mathbb{Z}$. If $\beta + i\omega \in \mathcal{W}_{i,k}$ for $k \in \mathbb{Z}$, the point*

$$\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(i,k)}) \in \text{Stab}_n(X/Y),$$

in Lemma 4.3 gives the point of \overline{U}_X .

Proof. First assume there exists $\sigma \in \overline{U}_X$ with $\mathcal{Z}_X(\sigma) = \beta + i\omega \in \mathcal{W}_i$. It is easy to see that $\mathcal{O}_{C_i}(k-1)$ is stable with respect to all $\sigma' \in U_X$ and $k \in \mathbb{Z}$. By the comment in [5] after [5, Proposition 8.1], $\mathcal{O}_{C_i}(k-1)$ is at least semistable in σ . This implies

$$Z_{(\beta,\omega)}(\mathcal{O}_{C_i}(k-1)) = -k + \beta \cdot C_i \neq 0,$$

for all $k \in \mathbb{Z}$. Therefore we have $\beta \cdot C_i \notin \mathbb{Z}$.

Conversely assume that $\beta \cdot C_i \in (k-1, k)$ for some $k \in \mathbb{Z}$. In this case, we have the stability condition

$$\sigma := \sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(i,k)}),$$

constructed in Lemma 4.3. Therefore it is enough to check $\sigma \in \overline{U}_X$. We leave it to the next Lemma, and give the proof in Section 6, because it requires some technical arguments. \square

Lemma 4.5 $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(i,k)})$, constructed in Lemma 4.3 is contained in \overline{U}_X .

Now we give the following definition.

Definition 4.6 We define $\partial U_X(i, k)$ for $1 \leq i \leq N$ and $k \in \mathbb{Z}$ by

$$\partial U_X(i, k) = \{(Z_{(\beta,\omega)}, \mathcal{A}_{(i,k)}) \in \overline{U}_X \mid \beta + i\omega \in \mathcal{W}_{i,k}\}.$$

4.3 Other regions by Fourier-Mukai transforms

Here we construct other regions using $\text{FM}(X)$ defined in Definition 3.13. Let us take $u = (W, \Phi) \in \text{FM}(X)$. Recall that Φ gives an equivalence $\Phi: D_{W/Y} \rightarrow D_{X/Y}$. Then for any $\sigma = (Z, \mathcal{P}) \in \text{Stab}(W/Y)$, one gets the stability condition $\Phi_*\sigma = (Z', \mathcal{P}')$ by

$$Z'(E) := Z \circ \Phi^{-1}(E), \quad \mathcal{P}'(E) := \Phi\mathcal{P}(\phi),$$

for $E \in \mathcal{D}_{X/Y}$ and $\phi \in \mathbb{R}$. Obviously, the map

$$\Phi_*: \text{Stab}(W/Y) \longrightarrow \text{Stab}(X/Y)$$

gives a homeomorphism. Furthermore by Lemma 3.14 (ii), Φ_* restricts to the homeomorphism,

$$\Phi_*: \text{Stab}_n(W/Y) \longrightarrow \text{Stab}_n(X/Y).$$

Definition 4.7 For $u = (W, \Phi) \in \text{FM}(X)$, we define the open set $U(W, \Phi)$ in $\text{Stab}_n(X/Y)$ to be

$$U(W, \Phi) := \Phi_*U_W \subset \text{Stab}_n(X/Y),$$

and the map $\phi^u: N^1(W/Y)_{\mathbb{C}} \rightarrow N^1(X/Y)_{\mathbb{C}}$ to be

$$\phi^u(D) = \text{ch}_1 \Phi(\mathcal{O}_W) + \phi_*D.$$

Here ϕ is the birational map $\phi: W \dashrightarrow X$.

We have introduced the map ϕ^u because the following lemma holds.

Lemma 4.8 For $u = (W, \Phi) \in \text{FM}(X)$, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Stab}_n(W/Y) & \xrightarrow{\Phi_*} & \text{Stab}_n(X/Y) \\ \mathcal{Z}_W \downarrow & & \downarrow \mathcal{Z}_X \\ N^1(W/Y)_{\mathbb{C}} & \xrightarrow{\phi^u} & N^1(X/Y)_{\mathbb{C}}. \end{array}$$

Proof. Let us take $\sigma = (Z_{(\beta, \omega)}, \mathcal{P}) \in \text{Stab}_n(W/Y)$, and $\mathcal{L} \in \text{Pic}(W)_{\mathbb{C}}$ with $\text{ch}_1(\mathcal{L}) = \beta + i\omega$. Then for $F \in \mathcal{D}_{W/Y}$, the value $Z_{(\beta, \omega)}(F)$ is written as $-\chi(\mathcal{L}, F)$ by Riemann-Roch theorem. Therefore for $E \in \mathcal{D}_{X/Y}$, one calculates $Z_{(\beta, \omega)}\Phi^{-1}(E)$ as follows,

$$\begin{aligned} Z_{(\beta, \omega)}\Phi^{-1}(E) &= -\chi(\mathcal{L}, \Phi^{-1}(E)) \\ &= -\chi(\Phi(\mathcal{L}), E) \\ &= -\text{ch}_0 \Phi(\mathcal{L}) \text{ch}_3 E + \text{ch}_1 \Phi(\mathcal{L}) \text{ch}_2 E \\ &= -\text{ch}_3 E + (\phi_* \text{ch}_1 \mathcal{L} + \text{ch}_1 \Phi(\mathcal{O}_W)) \text{ch}_2 E \\ &= -\text{ch}_3 E + \phi^u(\beta + i\omega) \text{ch}_2 E. \end{aligned}$$

Here the fourth isomorphism follows from Lemma 3.15. The above equality exactly means $\mathcal{Z}_X \circ \Phi_* = \phi^u \circ \mathcal{Z}_W$. \square

We have the following corollary.

Corollary 4.9 \mathcal{Z}_X restricts to give a homeomorphism between $U(W, \Phi)$ and $\phi_*A(W/Y)_{\mathbb{C}}$.

4.4 Chambers in $\text{Stab}_n(X/Y)$

Here we investigate the relationship of the regions $U(W, \Phi)$.

Proposition 4.10 *The regions $U(W, \Phi)$ satisfy the following:*

(i) $U(W, \Phi) \cap U(W', \Phi') \neq \emptyset$ if and only if there exists $\mathcal{L} \in \text{Pic}(W)$ such that

$$W \cong W' \quad \text{and} \quad \Phi^{-1} \circ \Phi' \cong \otimes \mathcal{L}.$$

(ii) $U(W, \Phi) \cap U(W', \Phi') = \emptyset$ but $\overline{U}(W, \Phi) \cap \overline{U}(W', \Phi') \neq \emptyset$ in a codimension one wall if and only if $W \dashrightarrow W'$ is a flop at a single rational curve, and we have

$$\Phi'^{-1} \circ \Phi \cong \otimes \mathcal{L}' \circ \Phi'' \circ \otimes \mathcal{L}$$

with Φ'' a standard equivalence

$$\Phi'': D^b(W) \longrightarrow D^b(W'),$$

and $\mathcal{L} \in \text{Pic}(W)$, $\mathcal{L}' \in \text{Pic}(W')$. Moreover all points in a codimension one wall of $\overline{U}(W, \Phi)$ are contained in $\overline{U}(W', \Phi')$ with (W', Φ') as above.

Proof. (i) We may assume $W' = X$ and $\Phi' = \text{id}$. Suppose $U(W, \Phi) \cap U(X, \text{id}) \neq \emptyset$. Then Φ takes $\text{Coh}_C(W)$ to $\text{Coh}_C(X)$. Note that simple objects of these categories are skyscraper sheaves. Hence for $p \in W$, we have $\Phi(\mathcal{O}_p) \cong \mathcal{O}_{\phi(p)}$ for some $\phi(p) \in X$. Since Φ is Fourier-Mukai type, ϕ gives an isomorphism $W \cong X$ and $\Phi \cong \otimes \mathcal{L}$ for some $\mathcal{L} \in \text{Pic}(X)$. (cf. [21, Theorem 3.2]).

(ii) Again we may assume $W' = X$ and $\Phi' = \text{id}$. First let us consider a flop $\phi_i: X_i^\dagger \dashrightarrow X$ as in the diagram (1). Let us consider the standard equivalence,

$$\Phi_i: D^b(X_i^\dagger) \longrightarrow D^b(X).$$

Then recall that Φ_i takes ${}^{-1}\text{Per}(X_i^\dagger/Y_i)$ to ${}^0\text{Per}(X/Y_i)$, by Theorem 3.11. Combined with Lemma 3.8 (ii), Φ_i restricts to give the equivalence:

$$\Phi_i: \left({}^0\text{Per}(X_i^\dagger/Y_i) \cap \mathcal{D}_{X_i^\dagger/Y} \right) \otimes_{\mathcal{O}_{X_i^\dagger}} (D_i^\dagger) \longrightarrow {}^0\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y}.$$

Here D_i^\dagger is a divisor on X_i^\dagger with $D_i^\dagger \cdot C_i^\dagger = 1$, and $D_i^\dagger \cdot C_j^\dagger = 0$ for other exceptional curves C_j^\dagger , $j \neq i$.

By Lemma 4.4, both sides appear as t-structures of some points in codimension one boundaries of $U_{X_i^\dagger}$ and U_X respectively. On the other hand if $\beta \in N^1(X_i^\dagger/Y)$ satisfies $\beta \cdot C_i^\dagger \in (0, 1)$, then $\phi_{i*}\beta \cdot C_i \in (-1, 0)$. Therefore Lemma 4.4 together with Lemma 4.8 gives the homeomorphism,

$$\Phi_{i*}: \partial U_{X_i^\dagger}(i, 1) \longrightarrow \partial U_X(i, 0).$$

Here recall that $\partial U_X(i, k)$ is defined in Definition 4.6. Similarly, we have a homeomorphism,

$$(\otimes \mathcal{O}_X(D_i))_*: \partial U_X(i, k) \longrightarrow \partial U_X(i, k+1).$$

Combining these, for $\mathcal{L} \in \text{Pic}(X_i^\dagger)$ and $\mathcal{L}' \in \text{Pic}(X)$ there exists a homeomorphism,

$$(\otimes \mathcal{L}' \circ \Phi_i \circ \otimes \mathcal{L})_*: \partial U_{X_i^\dagger}(i, l) \longrightarrow \partial U_X(i, k),$$

for some k and l , and conversely for all $k \in \mathbb{Z}$, there exist $\mathcal{L} \in \text{Pic}(X_i^\dagger)$ and $\mathcal{L}' \in \text{Pic}(X)$ which gives the homeomorphism as above. This gives the “if” part and the last statement.

Conversely, assume that $\overline{U}(W, \Phi)$ and \overline{U}_X intersects at a codimension one boundary, but $U(W, \Phi) \cap U_X = \emptyset$. Then, because $\text{Stab}_n(X/Y)$ is a complex manifold, there exists a pair $(X_i^\dagger, \Phi^\dagger)$ with $\Phi^\dagger = \otimes \mathcal{L}' \circ \Phi_i \circ \otimes \mathcal{L}$ given as above, such that

$$U(W, \Phi) \cap U(X_i^\dagger, \Phi^\dagger) \neq \emptyset.$$

Then (i) implies $W \cong X_i^\dagger$ and Φ^\dagger is given as $\otimes \mathcal{L}' \circ \Phi_i \circ \otimes \mathcal{L}''$ for some $\mathcal{L}'' \in \text{Pic}(W)$. \square

4.5 Proof of Theorem 1.1

Now we give the proof of Theorem 1.1.

Theorem 4.11 *We have a disjoint union of chambers:*

$$\mathcal{M} := \bigcup_{(W, \Phi) \in \text{FM}(X)} U(W, \Phi) \subset \text{Stab}_n^\circ(X/Y),$$

such that two chambers are either equal or disjoint. Moreover we have $\overline{\mathcal{M}} = \text{Stab}_n^\circ(X/Y)$.

Proof. By the definition, $(W, \Phi) \in \text{FM}(X)$ is obtained as a successive sequence of tensoring line bundles and standard equivalences. Therefore Proposition 4.10 implies $U(W, \Phi)$ are contained in the same connected component, i.e.

$$\mathcal{M} \subset \text{Stab}_n^\circ(X/Y).$$

Also by Proposition 4.10 (i), $U(W, \Phi) \cap U(W', \Phi') \neq \emptyset$ implies $U(W, \Phi) = U(W', \Phi')$. It remains to show $\overline{\mathcal{M}} = \text{Stab}_n^\circ(X/Y)$. Take $\sigma \in \text{Stab}_n^\circ(X/Y)$ and $\sigma_0 \in U_X$. Let

$$\gamma: [0, 1] \longrightarrow \text{Stab}_n^\circ(X/Y),$$

be a path such that $\gamma(0) = \sigma_0$ and $\gamma(1) = \sigma$. Note that by Theorem 3.3, the chambers in $N^1(X/Y)_\mathbb{C}$ given by $\phi_* A(W/Y)_\mathbb{C}$ for crepant small resolutions $g: W \rightarrow Y$ are finite. Because the map \mathcal{Z}_X is a local homeomorphism by Theorem 2.4, we may assume $\mathcal{Z}_X(\gamma)$ passes through only codimension one walls of these chambers by deforming γ if necessary. Then there exists a sequence,

$$0 < t_1 < t_2 \cdots < t_{n-1} < 1,$$

such that $\mathcal{Z}_X(\gamma(t_k))$ is contained in one of the codimension one walls, and $\mathcal{Z}_X(\gamma((t_{k-1}, t_k)))$ are contained in the chambers. Then $\gamma((0, t_1))$ is contained in U_X and by Proposition 4.10 (ii), $\gamma((t_1, t_2))$ is contained in some $U(W, \Phi)$ with $(W, \Phi) \in \text{FM}(X)$. Repeating this argument, we have $\gamma(t) \in \overline{\mathcal{M}}$ for all $t \in [0, 1]$. In particular, $\sigma = \gamma(1) \in \overline{\mathcal{M}}$. \square

5 Stability conditions and covering spaces

In this section, we give the proof of Theorem 1.2.

5.1 Root lattices and Weyl reflections

First we introduce the root lattice associated to the exceptional locus of $f: X \rightarrow Y$.

Definition 5.1 We define the \mathbb{Z} -module Λ_f by

$$\Lambda_f := \bigoplus_{i=1}^N \mathbb{Z}e_i,$$

and introduce the pairing $(\cdot, \cdot): \Lambda_f \times \Lambda_f \rightarrow \mathbb{Z}$ by $(e_i, e_i) = -2$, $(e_i, e_j) = 1$ if $i \neq j$ and $C_i \cap C_j \neq \emptyset$, and $(e_i, e_j) = 0$ if otherwise.

It is well known that the pairing (\cdot, \cdot) on Λ_f is negative definite. We define the sets $V(\Lambda_f)$, $S(\Lambda_f)$ by

$$V(\Lambda_f) := \{v \in \Lambda_f \mid (v, v) = -2\}, \quad S(\Lambda_f) = \{e_i \mid 1 \leq i \leq N\}.$$

An element $v \in V(\Lambda_f)$ is called a *root*, and $e_i \in S(\Lambda_f)$ is called a *simple root*. It is well-known that any element $v \in V(\Lambda_f)$ is written as

$$v = \sum_{i=1}^N r_i e_i,$$

with $r_i \in \mathbb{Z}_{\geq 0}$ for all i , or $r_i \in \mathbb{Z}_{\leq 0}$ for all i . For $v \in V(\Lambda_f)$, we associate the reflection,

$$r_v: \Lambda_f \ni x \mapsto x + (v, x)v \in \Lambda_f.$$

It is well-known that the set $V(\Lambda_f)$ is obtained by applying reflections $\{r_{e_i}\}_{1 \leq i \leq N}$ to the set $S(\Lambda_f)$. Note that we have a natural identification of vector spaces,

$$\Lambda_f \otimes_{\mathbb{Z}} \mathbb{R} \cong N_1(X/Y),$$

by identifying e_i with C_i . Then the dual of r_v gives the reflection,

$$r_v^*: N^1(X/Y) \longrightarrow N^1(X/Y),$$

and is called the *Weyl reflection*. Let us consider a partial resolution $X \xrightarrow{f'} Y' \rightarrow Y$. Then there is a natural inclusion $\Lambda_{f'} \hookrightarrow \Lambda_f$, and it is obvious that

$$V(\Lambda_{f'}) = V(\Lambda_f) \cap \Lambda_{f'}.$$

Let $g: W \rightarrow Y$ be another crepant small resolution. Then the dual graphs of the exceptional locus of f and g are identified, since they are identified under simple flops. (In our case we flops at single rational curves since Y is complete, thus this statement is obvious. However we note that the explicit constructions of flops [24, Theorem 2.4] would also imply this without the assumption that Y is complete.) Under the assumption that X_0 is smooth, it is possible to choose the identification of the dual graphs canonically (does not depend on a choice of a decomposition of $W \dashrightarrow X$ into flops) because the birational map $W \dashrightarrow X$ induces an isomorphism between minimal resolutions, $g^{-1}(Y_0) \xrightarrow{\cong} X_0$. Hence we have an identification, $\psi: \Lambda_g \cong \Lambda_f$ which preserves the set of simple roots. By taking its dual, we obtain an isomorphism,

$$\psi^*: N^1(X/Y) \longrightarrow N^1(W/Y).$$

By composing with the strict transform, we get the isomorphism,

$$\phi_* \circ \psi^*: N^1(X/Y) \longrightarrow N^1(W/Y) \longrightarrow N^1(X/Y).$$

If $\phi: W \dashrightarrow X$ is a flop at C_i , the above map is calculated in [28, Theorem 6.3], and it coincides with $r_{e_i}^*$. Here we note that in [28, Theorem 6.3], the assumption that a smooth surface $C \subset X_0$ with $(C)_{X_0}^2 = -2$ exists is needed. Combined with Lemma 4.8, we obtain the following.

Lemma 5.2 *We have the map,*

$$\Theta: \text{FM}(X) \ni u \longmapsto \phi^u \circ \psi^* \in \text{Aff}(N^1(X/Y)_{\mathbb{C}}).$$

Here the right hand side is the group of affine transformations on $N^1(X/Y)_{\mathbb{C}}$. Its image coincides with the subgroup generated by Weyl reflections with respect to simple roots of Λ_f and translations by the lattice $\text{Pic}(X) \subset N^1(X/Y)$.

5.2 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we introduce some notations. First note that the following set

$$G := \{(W, \Phi) \in \text{FM}(X) \mid W = X\},$$

is a subgroup of the group of autoequivalences of $D^b(X)$. Then the following map

$$G \ni (X, \Phi) \longmapsto \text{ch}_1 \Phi(\mathcal{O}_X) \in N^1(X/Y),$$

is a group homomorphism, by the same argument of the proof of Lemma 3.15. We define $G^\circ \subset G$ to be the kernel of the above map. Next for $v \in V(\Lambda_f)$, define $H_v \subset N^1(X/Y)_{\mathbb{C}}$ to be

$$H_v := \{\beta + i\omega \in N^1(X/Y)_{\mathbb{C}} \mid (\beta + i\omega)v \in \mathbb{Z}\}.$$

Now we prove Theorem 1.2.

Theorem 5.3 *We have the following map:*

$$\mathcal{Z}_X: \text{Stab}_n^\circ(X/Y) \longrightarrow N^1(X/Y)_{\mathbb{C}} \setminus \bigcup_{v \in V(\Lambda_f)} H_v.$$

It is a regular covering map with Galois group equal to G° .

Proof. We divide the proof into 3 steps.

Step 1 *We have $\text{Im}(\mathcal{Z}_X) \subset N^1(X/Y)_{\mathbb{C}} \setminus \bigcup_{v \in V(\Lambda_f)} H_v$.*

Proof. Note that for $u = (W, \Phi) \in \text{FM}(X)$ with $g: W \rightarrow Y$, ϕ^u restricts to an isomorphism,

$$\phi^u: N^1(W/Y)_{\mathbb{C}} \setminus \bigcup_{v \in V(\Lambda_g)} H_v \longrightarrow N^1(X/Y)_{\mathbb{C}} \setminus \bigcup_{v \in V(\Lambda_f)} H_v.$$

Therefore by Theorem 4.11 and Lemma 4.8, it suffices to show that for $\sigma = (Z_{(\beta, \omega)}, \mathcal{P}) \in \overline{U}_X$, we have $(\beta + i\omega)v \notin \mathbb{Z}$ for all $v \in V(\Lambda_f)$. First from Proposition 4.4, we have $(\beta + i\omega)v \notin \mathbb{Z}$ for simple roots v . Next suppose ω is written as $\omega = f'^* \omega'$ for a partial resolution $X \xrightarrow{f'} Y' \rightarrow Y$, and $\omega' \in A(Y'/Y)$. Then $(\beta + i\omega)v$ could be an integer only if

$$v \in V(\Lambda_f) \cap \Lambda_{f'} = V(\Lambda_{f'}).$$

Take $v \in V(\Lambda_{f'})$. Then by the argument of Lemma 5.2, there exist flops over Y' ,

$$\phi: W = X^n \dashrightarrow X^{n-1} \dashrightarrow \dots \dashrightarrow X^0 = X,$$

and a simple root $v' \in S(\Lambda_{g'})$ such that $\phi_* H_{v'} = H_v$. Here g' is a structure morphism $g': W \rightarrow Y'$. Note that

$$\omega \in f'^* A(Y'/Y) \subset \phi_* \overline{A}(W/Y).$$

Using Theorem 4.11 and the fact that \mathcal{Z}_X is a local isomorphism, we can find a region $U(W, \Phi)$ such that $\sigma \in \overline{U}(W, \Phi)$. Since $\mathcal{Z}_W(\overline{U}_W) \subset N^1(W/Y)_{\mathbb{C}} \setminus H_{v'}$, Proposition 4.8 implies

$$\beta + i\omega = \mathcal{Z}_X(\sigma) \in \mathcal{Z}_X(\overline{U}(W, \Phi)) \subset N^1(X/Y)_{\mathbb{C}} \setminus H_v.$$

This implies $(\beta + i\omega)v \notin \mathbb{Z}$. \square

Step 2 *The map \mathcal{Z}_X is surjective.*

Proof. By Lemma 4.8, it suffices to show the surjectivity on $\overline{A}(X/Y)_{\mathbb{C}} \setminus \bigcup_{v \in V(\Lambda_f)} H_v$. Let us take $\beta + i\omega \in \overline{A}(X/Y)_{\mathbb{C}}$ and suppose $\omega = f'^* \omega'$ for a partial resolution $X \xrightarrow{f'} Y' \rightarrow Y$, i.e. $\beta + i\omega$ lies in the wall $\mathcal{W} := N^1(X/Y) \oplus i f'^* A(Y'/Y)$. For a root $v \in V(\Lambda_{f'})$, let \overline{H}_v be

$$\overline{H}_v := \{\beta \in N^1(X/Y') \mid \beta \cdot v \in \mathbb{Z}\} \subset N^1(X/Y').$$

\overline{H}_v is a real codimension one hypersurface and we have the fiber space structure

$$\Pi: \mathcal{W} \setminus \bigcup_{v \in V(\Lambda_f)} H_v \longrightarrow N^1(X/Y') \setminus \bigcup_{v \in V(\Lambda_{f'})} \overline{H}_v,$$

with fiber $f'^* A(Y'/Y)_{\mathbb{C}}$. Let $C(X/Y')$ be one of the connected component of the right hand side, which contains the following set,

$$\{\beta \in N^1(X/Y') \mid -\varepsilon < \beta \cdot v < 0 \text{ for all } v \in S(\Lambda_{f'})\},$$

for sufficiently small $0 < \varepsilon \ll 1$. We are now going to argue that it suffices to check the surjectivity on $\Pi^{-1}(C(X/Y'))$.

Let \mathcal{G} be the group of Weyl reflections generated by the reflections by the simple roots $S(\Lambda_{f'})$, acting on $N^1(X/Y')$. Let $\text{Pic}(X/Y') \subset N^1(X/Y')$ be the lattice consisting of numerical classes of line bundles. Then the action of \mathcal{G} preserves $\text{Pic}(X/Y')$. Hence we can consider the semi-direct product $\mathcal{G} \ltimes \text{Pic}(X/Y')$ with its action on $N^1(X/Y') \setminus \bigcup_{v \in V(\Lambda_{f'})} \overline{H}_v$ given by the following formula:

$$(g, \mathcal{L}) * D = g(D) + \mathcal{L}.$$

Here $g \in \mathcal{G}$, $\mathcal{L} \in \text{Pic}(X/Y')$ and $D \in N^1(X/Y')$. This action also induces the action on the set of connected components of $N^1(X/Y') \setminus \bigcup_{v \in V(\Lambda_{f'})} \overline{H}_v$, which can be easily checked to be transitive.

On the other hand, let us use Lemma 5.2. Then for each $(\phi, \mathcal{L}) \in \mathcal{G} \ltimes \text{Pic}(X/Y')$, we can find $(W, \Phi) \in \text{FM}(X)$ such that

- A successive sequence

$$W = X^n \dashrightarrow X^{n-1} \dashrightarrow \dots \dashrightarrow X^1 \dashrightarrow X^0 = X,$$

of Definition 3.13 is defined over Y' , i.e. the structure morphism $X^j \rightarrow Y$ factors as $X^j \rightarrow Y' \rightarrow Y$. In particular there exists a morphism $g': W \rightarrow Y'$ and $\Theta(W, \Phi)$ preserves the wall \mathcal{W} .

- We have $\Pi \circ \Theta(W, \Phi)|_{\mathcal{W}} = (\phi, \mathcal{L}) \circ \Pi$.

Also note that for any lattice isomorphism $\psi: \Lambda_{g'} \rightarrow \Lambda_{f'}$ which preserve the simple roots, the induced morphism $\psi^*: N^1(X/Y') \rightarrow N^1(W/Y')$ takes $C(X/Y')$ to $C(W/Y')$. Therefore using Lemma 4.8, we may assume $\Pi(\beta + i\omega)$ lies in $C(X/Y')$. In this region, let us take the pair:

$$\sigma := (Z_{(\beta, \omega)}, {}^0\text{Per}(X/Y') \cap \mathcal{D}_{X/Y}).$$

By the same proof as in Lemma 4.3 and Proposition 4.4, we can conclude $\sigma \in \text{Stab}_n^\circ(X/Y)$. \square

Step 3 *The map \mathcal{Z}_X is a regular covering map with Galois group G° .*

Proof. First for $\sigma \in U_X$ and $u = (X, \Phi) \in G^\circ$, one has

$$\Phi_* u \in U(X, \Phi) \subset \text{Stab}_n^\circ(X/Y).$$

Therefore the action of G° on the set of stability conditions preserve the connected component $\text{Stab}_n^\circ(X/Y)$. Moreover by the definition of G° , one has $\phi^u = \text{id}$. Therefore Lemma 4.8 implies G° acts on $\text{Stab}_n^\circ(X/Y)$ as deck transformations. This action is also faithful. In fact, assume that for $(X, \Phi) \in G^\circ$, one has $\Phi_* \sigma = \sigma$ for some $\sigma \in \text{Stab}_n^\circ(X/Y)$. By Theorem 4.11, we may assume $\sigma \in U(W, \Phi')$ for some $(W, \Phi') \in \text{FM}(X)$. Then

$$U(W, \Phi') \cap U(W, \Phi \circ \Phi') \neq \emptyset.$$

By Lemma 4.10 (i), Φ is given by tensoring line bundle. Because $\text{ch}_1 \Phi(\mathcal{O}_X) = 0$, we have $\Phi = \text{id}$.

Next let us take $\sigma, \sigma' \in \text{Stab}_n^\circ(X/Y)$ such that $\mathcal{Z}_X(\sigma) = \mathcal{Z}_X(\sigma')$. By Theorem 4.11, there exist some regions $U(W, \Phi), U(W, \Phi')$ such that $\sigma \in \overline{U}(W, \Phi)$ and $\sigma' \in \overline{U}(W, \Phi')$. Let us choose a sequence $\{\sigma_k\}_{k=1}^\infty \subset U(W, \Phi)$ which converges to σ . Then there exists another sequence $\{\sigma'_k\}_{k=1}^\infty \subset U(W, \Phi')$ which converges to σ' and $\mathcal{Z}_X(\sigma_k) = \mathcal{Z}_X(\sigma'_k)$. Let us take $\mathcal{L} \in \text{Pic}(W)$ to be

$$\mathcal{L} = \det \Phi'^{-1}(\mathcal{O}_X) \otimes \{\det \Phi^{-1}(\mathcal{O}_X)\}^{-1}.$$

Then by Lemma 6.1 below, the functor

$$\Phi'' := \Phi' \circ \otimes \mathcal{L} \circ \Phi^{-1}: D^b(X) \rightarrow D^b(X),$$

satisfies $\Phi''(\mathcal{O}_X)|_{X^\circ} \cong \mathcal{O}_{X^\circ}$, where $X^\circ = X \setminus C$. Because $C \subset X$ is codimension two, this implies $\det \Phi''(\mathcal{O}_X) \cong \mathcal{O}_X$, hence $\text{ch}_1 \Phi''(\mathcal{O}_X) = 0$. Therefore $(X, \Phi'') \in G^\circ$ and

$$\mathcal{Z}_X(\Phi''_* (\sigma_k)) = \mathcal{Z}_X(\sigma_k) = \mathcal{Z}_X(\sigma'_k).$$

Note that Φ''_* takes $U(W, \Phi)$ to $U(W, \Phi')$, thus $\Phi''_*(\sigma_k)$ is contained in $U(W, \Phi')$. Since \mathcal{Z}_X is injective on $U(W, \Phi')$, it follows that

$$\Phi''_*(\sigma_k) = \sigma'_k.$$

Taking the limit $k \rightarrow \infty$, we get $\Phi''_*(\sigma) = \sigma'$. This shows \mathcal{Z}_X is a regular covering map with Galois group G° . \square

Theorem 5.3 provides some information of the group of autoequivalences of $\mathcal{D}_{X/Y}$. Let $\text{Aut}^\circ(\mathcal{D}_{X/Y})$ be the group of autoequivalences Φ of $\mathcal{D}_{X/Y}$, such that

- Φ preserves the class $[\mathcal{O}_x] \in K(\mathcal{D}_{X/Y})$
- Φ preserves the connected component $\text{Stab}_n^\circ(X/Y)$.

Then we have the following.

Corollary 5.4 (i) *We have the group isomorphism*

$$G = \{(W, \Phi) \in \text{FM}^\circ(X) \mid W = X\} \cong \text{Aut}^\circ(\mathcal{D}_{X/Y}).$$

(ii) *There exists the following exact sequence:*

$$\pi_1(N^1(X/Y)_{\mathbb{C}} \setminus \cup_{v \in V(\Lambda_f)} H_v) \longrightarrow \text{Aut}^\circ(\mathcal{D}_{X/Y}) \longrightarrow \text{Pic}(X) \longrightarrow 0.$$

Here the last map is defined by taking Φ to $\det \Phi(\mathcal{O}_X)$.

Proof. (i) Let us take $\Phi \in \text{Aut}^\circ(\mathcal{D}_{X/Y})$ and $\sigma \in U_X$. Then $\Phi_*\sigma \in \text{Stab}_n^\circ(X/Y)$. By Theorem 4.11, $\Phi_*\sigma \in \overline{U}(W, \Phi')$ for some $(W, \Phi') \in \text{FM}(X)$. By deforming σ if necessary, we may assume $\Phi_*\sigma \in U(W, \Phi')$. Then we have

$$\Phi_*\sigma \in U(X, \Phi) \cap U(W, \Phi') \neq \emptyset.$$

Thus by Proposition 4.10 (i), we have $W \cong X$ and $\Phi \cong \otimes \mathcal{L} \circ \Phi'$ for some $\mathcal{L} \in \text{Pic}(X)$. Therefore $(X, \Phi) \in \text{FM}(X)$.

(ii) This follows from (i) and Theorem 5.3. \square

Remark 5.5 If we knew $\text{Stab}_n^\circ(X/Y)$ is simply connected, we have the exact sequence:

$$0 \longrightarrow \pi_1(N^1(X/Y)_{\mathbb{C}} \setminus \cup_{v \in V(\Lambda_f)} H_v) \longrightarrow \text{Aut}^\circ(\mathcal{D}_{X/Y}) \longrightarrow \text{Pic}(X) \longrightarrow 0.$$

Example

Let $Y = \text{Spec } \mathbb{C}[[x, y, z, w]]/(xy - zw)$ and $f: X \rightarrow Y$ be the blowing up at the ideal (x, z) as in the introduction. Then Theorem 5.3 describes $\text{Stab}_n^\circ(X/Y)$ as a covering map,

$$\text{Stab}_n^\circ(X/Y) \longrightarrow \mathbb{C} \setminus \mathbb{Z}.$$

In this case $\text{Pic}(X) = \mathbb{Z}$, and the action of $n \in \text{Pic}(X)$ on $N^1(X/Y)_{\mathbb{C}} = \mathbb{C}$ is given by the translation, $\beta + i\omega \mapsto n + \beta + i\omega$. Therefore Corollary 5.4 implies,

$$\begin{aligned} \text{Stab}_n^\circ(X/Y) / \text{Aut}^\circ(\mathcal{D}_{X/Y}) &\cong (\mathbb{C} \setminus \mathbb{Z}) / \mathbb{Z} \\ &\cong \mathbb{P}^1 \setminus \{\text{three points}\}. \end{aligned}$$

Thus we have obtained the same picture as in [1, Figure 2].

Remark 5.6 If we don't assume the existence of a smooth surface X_0 , the description may change. For example, there is an example of a crepant small resolution $f: X \rightarrow Y$ which contracts a $(1, -3)$ -curve and its scheme theoretic fiber is non-reduced [28, Lemma 5.16]. In this case we have to delete extra rational points from $\mathbb{C} \setminus \mathbb{Z}$, according to the proof of Lemma 4.3. Thus Theorem 5.3 is false in this case. At this time, the author does not know how to describe $\text{Stab}_n^\circ(X/Y)$ in a beautiful way in a general situation involving such cases.

6 Some technical lemmas

In this section, we give the postponed proofs.

6.1 Proof of Lemma 3.8

Proof. (i) For simplicity we show the case of $p = 0$. The objects of ${}^0\text{Per}(\mathcal{D}_{X/Y_i})$ are contained in the right hand side by the definition. Take $E \in \text{Coh}_{\cup_{j \neq i} C_j}(X)$. Then $\mathbf{R}g_{i*}E \in D^b(Y_i)$ is a sheaf, hence Lemma 3.5 applied to $g_i: X \rightarrow Y_i$ implies $E \in {}^0\text{Per}(X/Y_i)$. Since E is also supported on C , E is contained in the right hand side.

Conversely let $\mathcal{B} \subset \mathcal{D}_{X/Y}$ the smallest extension closed subcategory which contains the left hand side. Let us take $A \in {}^0\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y}$. Then by Lemma 3.5(i), we have the distinguished triangle,

$$H^{-1}(A)[1] \longrightarrow A \longrightarrow H^0(A) \longrightarrow H^{-1}(A)[2].$$

The conditions of Lemma 3.5 applied to $g_i: X \rightarrow Y_i$ shows that $H^{-1}(A)[1]$ is also contained in ${}^0\text{Per}(X/Y_i)$. By Lemma 3.5(ii), one has $g_{i*}H^{-1}(A) = 0$. Therefore $H^{-1}(A)$ must be supported on C_i , hence

$$H^{-1}(A)[1] \in {}^0\text{Per}(\mathcal{D}_{X/Y_i}) \subset \mathcal{B}.$$

Therefore it is enough to check $H^0(A) \in \mathcal{B}$. Let $I_C \subset I_{C_i} \subset \mathcal{O}_X$ be the ideal sheaves which define C , C_i respectively. Since $H^0(A)$ is supported on C , it is annihilated by I_C^n for some $n \in \mathbb{N}$. We have the exact sequence,

$$0 \rightarrow I_{C_i}^n/I_C^n \rightarrow \mathcal{O}_X/I_C^n \rightarrow \mathcal{O}_X/I_{C_i}^n \rightarrow 0.$$

Note that $I_{C_i}^n/I_C^n$ is supported on $\cup_{j \neq i} C_j$ and $\mathcal{O}_X/I_{C_i}^n$ is supported on C_i . By tensoring $H^0(A)$, we get the exact sequence

$$0 \longrightarrow A_1 \longrightarrow H^0(A) \longrightarrow A_2 \longrightarrow 0,$$

where $A_2 = H^0(A) \otimes \mathcal{O}_X/I_{C_i}^n$ and A_1 is the image of the map $H^0(A) \otimes I_{C_i}^n/I_C^n \rightarrow H^0(A)$. Note that $A_1 \in \text{Coh}_{\cup_{j \neq i} C_j}(X) \subset \mathcal{B}$ and $A_2 \in \text{Coh}_{C_i}(X)$. On the other hand, since $R^1g_{i*}H^0(A) = 0$ by Lemma 3.5(ii), the long exact sequence associated to the above sequence gives $R^1g_{i*}A_2 = 0$. Therefore A_2 satisfies the conditions of Lemma 3.5, thus $A_2 \in {}^0\text{Per}(\mathcal{D}_{X/Y_i}) \subset \mathcal{B}$. Therefore by the exact sequence above, we have $H^0(A) \in \mathcal{B}$.

(ii) Note that the restriction of $g_i: X \rightarrow Y_i$ to X_0 is a minimal resolution of A_1 -singularity. Thus the scheme theoretic fiber of g_i is reduced. Hence by (i) and Proposition 3.6, the left hand side is the smallest extension closed subcategory of $\mathcal{D}_{X/Y}$ which contains

$$\text{Coh}_{\cup_{j \neq i} C_j}(X) \cup \{\mathcal{O}_{C_i}, \mathcal{O}_{C_i}(-1)[1]\}.$$

Similarly the right hand side is the smallest extension closed subcategory of $\mathcal{D}_{X/Y}$ which contains

$$\begin{aligned} & \text{Coh}_{\cup_{j \neq i} C_j}(X) \cup \{\mathcal{O}_{C_i}(-2)[1], \mathcal{O}_{C_i}(-1)\} \otimes \mathcal{O}_X(D_i) \\ &= \text{Coh}_{\cup_{j \neq i} C_j}(X) \cup \{\mathcal{O}_{C_i}(-1)[1], \mathcal{O}_{C_i}\}. \end{aligned}$$

Therefore both sides coincide.

(iii) Since the scheme theoretic fiber of g_i is reduced, this follows from (i) and Proposition 3.6.

□

6.2 Proof of Lemma 3.14

Proof. (i) First we check $[\mathcal{O}_x]$ does not depend on a choice of $x \in C$. Since $[\mathcal{O}_x] = [\mathcal{O}_{C_i}] - [\mathcal{O}_{C_i}(-1)]$ if $x \in C_i$, $[\mathcal{O}_x]$ does not depend on $x \in C_i$. Moreover since C is connected, one can conclude $[\mathcal{O}_x]$ does not depend on $x \in C$.

Next one has

$$\chi(\mathcal{O}_X, \mathcal{O}_x) = 1, \quad \chi(\mathcal{O}_X, \mathcal{O}_{C_i}(-1)) = 0, \quad \chi(\mathcal{O}_X(D_j), \mathcal{O}_{C_i}(-1)) = -\delta^{ij}.$$

Here $\delta^{ij} = 1$ if $i = j$ and zero if $i \neq j$. Therefore $\{[\mathcal{O}_x], [\mathcal{O}_{C_i}(-1)]\}_{1 \leq i \leq N}$ are linearly independent.

Finally, we check the set $\{[\mathcal{O}_x], [\mathcal{O}_{C_i}(-1)]\}_{1 \leq i \leq N}$ spans $K(\mathcal{D}_{X/Y})$. For $E \in \mathcal{D}_{X/Y}$, we can write $[E] = \sum (-1)^i [H^i(E)]$ in $K(\mathcal{D}_{X/Y})$. So we assume $E \in \text{Coh}_C(X)$. Then one has a filtration

$$E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that $F_j = E_j/E_{j-1}$ is a \mathcal{O}_{C_i} -module for some i . Since $[F_j]$ is written as a sum of $[\mathcal{O}_{x_i}]$ for a $x_i \in C_i$ and $[\mathcal{O}_{C_i}(-1)]$, $[E]$ is written as a linear combination of $\{[\mathcal{O}_x], [\mathcal{O}_{C_i}(-1)]\}_{1 \leq i \leq N}$.

(ii) By the definition of $\text{FM}(X)$, Φ is decomposed into tensoring line bundles and standard equivalences. Clearly tensoring a line bundle preserves the class $[\mathcal{O}_x]$. Also a standard equivalence preserve the class $[\mathcal{O}_x]$, because Φ takes point sheaves to point sheaves away from a flopped curve. \square

6.3 Proof of Lemma 3.15

First we introduce some notations, and prepare a lemma. For each crepant small resolution $g: W \rightarrow Y$, let $Y^\circ := Y \setminus \{0\}$, $W^\circ := W \setminus \text{Ex}(g)$ and $j_W: W^\circ \hookrightarrow W$ be the open immersion. Here $\text{Ex}(g)$ is the exceptional locus of g . Let us take $(W, \Phi) \in \text{FM}(X)$. Note that the birational map $\phi: W \dashrightarrow X$ gives an isomorphism $\phi^\circ: W^\circ \rightarrow X^\circ$. It seems the following lemma is well-known, but we give the proof for lack of reference.

Lemma 6.1 *There exists an equivalence $\Phi^\circ: D^b(W^\circ) \rightarrow D^b(X^\circ)$ such that the following diagram commutes,*

$$\begin{array}{ccc} D^b(W) & \xrightarrow{\Phi} & D^b(X) \\ j_W^* \downarrow & & \downarrow j_X^* \\ D^b(W^\circ) & \xrightarrow{\Phi^\circ} & D^b(X^\circ). \end{array}$$

More precisely, $\Phi^\circ(\mathcal{O}_{W^\circ})$ is a line bundle, and for $E \in D^b(W^\circ)$ we have

$$\Phi^\circ(E) \cong \phi_*^\circ E \otimes \Phi^\circ(\mathcal{O}_{W^\circ}).$$

Proof. Let $\mathcal{E} \in D^b(X \times W)$ be the kernel of Φ . Define \mathcal{E}° to be

$$\mathcal{E}^\circ := \mathcal{E}|_{X^\circ \times W^\circ} \in D^b(X^\circ \times W^\circ).$$

Then let Φ° be

$$\Phi^\circ := \Phi_{W^\circ \rightarrow X^\circ}^{\mathcal{E}^\circ}: D^b(W^\circ) \longrightarrow D^b(X^\circ).$$

We check that the above diagram commutes. Let p_X°, p_W° be the projections from $X^\circ \times W^\circ$ onto X° and W° , and $p'_X: X^\circ \times W \rightarrow X^\circ$ be the projection. For $E \in D^b(W)$, one has

$$\begin{aligned}
j_X^* \Phi(E) &\cong j_X^* \mathbf{R}p_{X*}(p_W^*(E) \otimes^{\mathbf{L}} \mathcal{E}) \\
&\cong \mathbf{R}p'_{X*}\{(p_W^*(E) \otimes^{\mathbf{L}} \mathcal{E})|_{X^\circ \times W}\} \\
&\cong \mathbf{R}p_{X*}^\circ\{(p_W^*(E) \otimes^{\mathbf{L}} \mathcal{E})|_{X^\circ \times W^\circ}\} \\
&\cong \mathbf{R}p_{X*}^\circ\{p_W^{\circ*} j_W^* E \otimes^{\mathbf{L}} \mathcal{E}^\circ\} \\
&\cong \Phi^\circ \circ j_W^*(E).
\end{aligned}$$

Here the second isomorphism follows from the base change formula for the diagram below,

$$\begin{array}{ccc}
X^\circ \times W & \longrightarrow & X \times W \\
p'_X \downarrow & & \downarrow p_X \\
X^\circ & \xrightarrow{j_X} & X,
\end{array}$$

and the third isomorphism follows because \mathcal{E} is supported on $X \times_Y W$.

Finally we check Φ° gives an equivalence. If Φ is given by tensoring $\mathcal{L} \in \text{Pic}(X)$, then Φ° is also given by tensoring $\mathcal{L}|_{X^\circ}$. If Φ is given by the standard equivalence, Φ° is given by ϕ_*° . Therefore in these cases, Φ° is an equivalence. Also if we take $(W', \Phi') \in \text{FM}(W)$, then it is easily checked that

$$\Phi^\circ \circ \Phi'^\circ \cong (\Phi \circ \Phi')^\circ: D^b(W'^\circ) \longrightarrow D^b(W^\circ).$$

Therefore by the definition of $\text{FM}(X)$, the functor Φ° gives an equivalence. Moreover by the description of Φ° for type I and II in Definition 3.13, for any $(W, \Phi) \in \text{FM}(X)$, Φ° is given by

$$\Phi^\circ(E) \cong \phi_*^\circ(E) \otimes \mathcal{L},$$

for some $\mathcal{L} \in \text{Pic}(X^\circ)$. Applying $E = \mathcal{O}_{W^\circ}$, one has $\mathcal{L} = \Phi^\circ(\mathcal{O}_{W^\circ})$. \square

Proof of Lemma 3.15.

For $E \in D^b(X)$, let $\det(E) \in \text{Pic}(X)$ be the determinant line bundle. Because $\text{ch}_1 E = \text{ch}_1 \det(E)$ and $\text{ch}_0 \Phi(\mathcal{O}_W) = 1$, it is sufficient to prove

$$\det \Phi(\mathcal{L}) \cong \det(\phi_* \mathcal{L} \otimes \Phi(\mathcal{O}_W)).$$

Since $C \subset X$ has codimension two, one has $j_{X*} j_X^* E \cong E$ for a vector bundle E on X . Hence it is enough to show the above formula on X° . On the other hand, by Lemma 6.1 we have

$$\begin{aligned}
\Phi(\mathcal{L})|_{X^\circ} &\cong \Phi^\circ(\mathcal{L}|_{W^\circ}) \\
&\cong \phi_*^\circ(\mathcal{L}|_{W^\circ}) \otimes \Phi^\circ(\mathcal{O}_{W^\circ}) \\
&\cong (\phi_* \mathcal{L} \otimes \Phi(\mathcal{O}_W))|_{X^\circ}.
\end{aligned}$$

Note that taking determinants and the restrictions to X° commute. Thus one has

$$\begin{aligned}
\det(\Phi(\mathcal{L}))|_{X^\circ} &\cong \det(\Phi(\mathcal{L})|_{X^\circ}) \\
&\cong \det((\phi_* \mathcal{L} \otimes \Phi(\mathcal{O}_W))|_{X^\circ}) \\
&\cong \det(\phi_* \mathcal{L} \otimes \Phi(\mathcal{O}_W))|_{X^\circ},
\end{aligned}$$

and the lemma follows. \square

6.4 Proof of Lemma 4.5

Before proving Lemma 4.5, we prepare some lemmas. First we give the following technical lemma on the perverse t-structures. Recall that by the definition, $\mathcal{A}_{(i,0)}$ is given by

$$\mathcal{A}_{(i,0)} = {}^0\text{Per}(X/Y_i) \cap \mathcal{D}_{X/Y}.$$

Lemma 6.2 (i) For any object $E \in \mathcal{A}_{(i,0)}$, the following sequence is an exact sequence in $\mathcal{A}_{(i,0)}$.

$$0 \longrightarrow H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0,$$

with $H^{-1}(E)[1] \in {}^0\text{Per}(\mathcal{D}_{X/Y_i})$.

(ii) For any object $E \in \text{Coh}_C(X)$, there exists an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,$$

in $\text{Coh}_C(X)$ such that

$$F \in \mathcal{A}_{(i,0)} \cap \text{Coh}_C(X), \quad G[1] \in {}^0\text{Per}(\mathcal{D}_{X/Y_i}).$$

Proof. (i) By the criterion of Lemma 3.5 applied to $g_i: X \rightarrow Y_i$, for $E \in \mathcal{A}_{(i,0)}$ the objects

$$H^{-1}(E)[1], \quad H^0(E)$$

are also contained in $\mathcal{A}_{(i,0)}$. Because $\mathcal{A}_{(i,0)}$ is a heart of a bounded t-structure on $\mathcal{D}_{X/Y}$, the distinguished triangle

$$H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E)$$

gives a short exact sequence in $\mathcal{A}_{(i,0)}$. By Lemma 3.5 (ii), we have $g_{i*}H^{-1}(E) = 0$. Therefore $H^{-1}(E)$ is supported on C_i .

(ii) For this proof, we refer to [12, Section 3]. In fact, according to [12, Section 3], the category ${}^0\text{Per}(X/Y_i)$ is obtained from $\text{Coh}(X)$ as a *tilting* [16] for a certain torsion pair $(\mathcal{T}_0, \mathcal{F}_0)$ on $\text{Coh}(X)$, with \mathcal{T}_0 a torsion part and \mathcal{F}_0 a free part. According to [12, Section 3], an object $F \in \mathcal{F}_0$ must satisfy $g_{i*}F = 0$. Therefore F is supported on C_i , and this shows the torsion theory $(\mathcal{T}_0, \mathcal{F}_0)$ induce the torsion pair $(\mathcal{T}'_0, \mathcal{F}'_0)$ on $\text{Coh}_C(X)$. Clearly corresponding tilting is $\mathcal{A}_{(i,0)}$. Therefore we can decompose $E \in \text{Coh}_C(X)$ into the exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0,$$

such that $T \in \mathcal{T}'_0$ and $F \in \mathcal{F}'_0$. By the construction of the tilting [16, Proposition 2.1], T and $F[1]$ are objects in $\mathcal{A}_{(i,0)}$. Because F must be supported on C_i , one has $F[1] \in {}^0\text{Per}(\mathcal{D}_{X/Y_i})$. \square

We use the assumption that X_0 is smooth in the following lemma.

Lemma 6.3 There exists a constant $K > 0$ such that if $E \in \text{Coh}_C(X)$ satisfies $\text{Hom}(E, E) = \mathbb{C}$, we have

$$\text{ch}_2(E) = \sum_{i=1}^N r_i C_i,$$

with $0 \leq r_i \leq K$.

Proof. By [17, Lemma 3.8], E is a $\mathcal{O}_{f^{-1}(0)}$ -module where $f^{-1}(0)$ is a scheme theoretic fiber of f at the closed point $0 \in Y$. In particular for a hyperplane section $0 \in Y_0 \subset Y$ and its pull back $X_0 = f^{-1}(Y_0)$, E is an \mathcal{O}_{X_0} -module. Let us regard E as an object of $\text{Coh}(X_0)$. Note that we have

$$\text{Hom}_X(E, E) = \text{Hom}_{X_0}(E, E) = \mathbb{C}.$$

Because we assume X_0 is smooth, we can apply Riemann-Roch theorem on X_0 , and get

$$\begin{aligned} -\text{ch}_1(E)^2 &= \sum (-1)^i \dim \text{Ext}_{X_0}^i(E, E) \\ &\leq 2. \end{aligned}$$

Here we have taken the chern character $\text{ch}_1(E)$ in $N_1(X_0/Y_0)$. It is well-known that the pairing

$$N_1(X_0/Y_0) \times N_1(X_0/Y_0) \ni (D_1, D_2) \longmapsto D_1 \cdot D_2 \in \mathbb{R},$$

is negative definite. Therefore there are only finite number of possibilities for $r_i \in \mathbb{Z}_{\geq 0}$ which satisfy

$$-\left(\sum_{i=1}^N r_i C_i\right)^2 \leq 2,$$

and this shows the lemma. \square

Let us take $K > 0$ as in Lemma 6.3, and define $\mathcal{S} \subset \text{Coh}_C(X)$ to be

$$\mathcal{S} := \{E \in \text{Coh}_C(X) \mid \text{ch}_2(E) = \sum_{i=1}^N r_i C_i, \text{ with } 0 \leq r_i \leq K\}.$$

Proof of Lemma 4.5.

For $n \in \mathbb{N}$, let ω_n be

$$\omega_n := \omega + \frac{1}{n} D_i \in A(X/Y).$$

Because ω_n is ample, one has the stability condition

$$\sigma_n := (Z_{(\beta, \omega_n)}, \text{Coh}_C(X)) \in U_X,$$

constructed in Lemma 4.1. It suffices to show σ_n converges to σ .

For $E \in \mathcal{A}_{(i,k)}$, we denote by $\phi(E) \in (0, 1]$ the phase with respect to the stability function $Z_{(\beta, \omega)}$ of Proposition 2.2. Also for a σ -semistable object $E \in \mathcal{D}_{X/Y}$, we use the same notation for its phase $\phi(E) \in \mathbb{R}$. Of course these two notions are compatible, i.e. if $E \in \mathcal{A}_{(i,k)}$ is σ -semistable, the above two phases $\phi(E)$ coincide. Similarly, we use the notation $\phi_n(E')$ for the phase of $E' \in \text{Coh}_C(X)$ with respect to $Z_{(\beta, \omega_n)}$ and σ_n -semistable object $E' \in \mathcal{D}_{X/Y}$.

Note that ω_n converges to ω . Hence by the topology on $\text{Stab}(X/Y)$ introduced in [5, Section 6], and using [5, Lemma 6.1], it suffices to show the following: for any $\varepsilon > 0$, there exists $M > 0$ such that $n > M$ implies,

$$\begin{aligned} &\text{for any } \sigma\text{-stable object } E \in \mathcal{A}_{(i,k)}, \text{ if } F_n, F'_n \in \mathcal{D}_{X/Y} \text{ are the semistable factors} \\ &\text{of } E \text{ in } \sigma_n \text{ whose phases are the largest and the smallest respectively, one has} \quad (3) \\ &|\phi_n(F_n) - \phi(E)| < \varepsilon, \quad |\phi_n(F'_n) - \phi(E)| < \varepsilon. \end{aligned}$$

For simplicity we treat the case of $k = 0$. Other cases are similarly discussed. We divide the proof into two steps.

Step 1 *There exists $M > 0$ such that (3) holds for any $n > M$ and any σ -stable object $E \in \mathcal{A}_{(\beta, \omega)}$ with $\phi(E) = 1$.*

Proof. By Lemma 3.8 (i), any object $E \in \mathcal{A}_{(i, 0)}$ is given by a successive extension of the objects in

$$\mathrm{Coh}_{\cup_{j \neq i} C_j}(X), \quad \mathcal{O}_{C_i}(-2)[1], \quad \mathcal{O}_{C_i}(-1).$$

Note that any sheaf $F \in \mathrm{Coh}_{\cup_{j \neq i} C_j} X$ with $\dim \mathrm{Supp}(F) = 1$ satisfies $\mathrm{Im} Z_{(\beta, \omega)}(F) > 0$ because $\omega \cdot C_j > 0$ for $j \neq i$. Therefore if $E \in \mathcal{A}_{(i, 0)}$ satisfies $\mathrm{Im} Z_{(\beta, \omega)}(E) = 0$, E is given by a successive extension of the following objects,

$$\{\mathcal{O}_x \mid x \in \cup_{j \neq i} C_j\}, \quad \mathcal{O}_{C_i}(-2)[1], \quad \mathcal{O}_{C_i}(-1).$$

Therefore any σ -stable object E of $\phi(E) = 1$ is one of the above objects. Note that these objects are also stable in σ_n . One can easily calculate $\phi_n(\mathcal{O}_x) = \phi(\mathcal{O}_x) = 1$ and

$$\lim_{n \rightarrow \infty} \phi_n(\mathcal{O}_{C_i}(-2)[1]) = \phi(\mathcal{O}_{C_i}(-2)[1]) = 1, \quad \lim_{n \rightarrow \infty} \phi_n(\mathcal{O}_{C_i}(-1)) = \phi(\mathcal{O}_{C_i}(-1)) = 1.$$

Therefore it is enough to choose M so that $n > M$ implies

$$|\phi_n(\mathcal{O}_{C_i}(-2)[1]) - \phi(\mathcal{O}_{C_i}(-2)[1])| < \varepsilon, \quad |\phi_n(\mathcal{O}_{C_i}(-1)) - \phi(\mathcal{O}_{C_i}(-1))| < \varepsilon. \quad \square$$

Step 2 *There exists $M' > 0$ such that (3) holds for any $n > M'$ and any σ -stable object $E \in \mathcal{A}_{(i, 0)}$ with $0 < \phi(E) < 1$.*

Proof. Let us take a σ -stable object $E \in \mathcal{A}_{(i, 0)}$ with $0 < \phi(E) < 1$. Then we have the exact sequence in $\mathcal{A}_{(i, 0)}$ by Lemma 6.2 (i),

$$0 \longrightarrow H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0.$$

By Lemma 6.2 (i), $H^{-1}(E)$ is supported on C_i . Therefore $\phi(H^{-1}(E)[1]) = 1$ and the stability of E implies $H^{-1}(E)[1] = 0$. Therefore we have

$$E \in \mathcal{A}_{(i, 0)} \cap \mathrm{Coh}_C(X).$$

Let $F_n \in \mathrm{Coh}_C(X)$ be the semistable factor of E in σ_n of the largest phase. We have the exact sequence in $\mathrm{Coh}_C(X)$,

$$0 \longrightarrow F_n \longrightarrow E \longrightarrow G_n \longrightarrow 0, \tag{4}$$

with

$$\phi_n(F_n) \geq \phi_n(E). \tag{5}$$

By Lemma 6.2 (ii), we have the exact sequence in $\mathrm{Coh}_C(X)$,

$$0 \longrightarrow H_n \longrightarrow F_n \longrightarrow I_n \longrightarrow 0, \tag{6}$$

such that

$$H_n \in \mathcal{A}_{(i, 0)} \cap \mathrm{Coh}_C(X), \quad I_n[1] \in {}^0\mathrm{Per}(\mathcal{D}_{X/Y_i}).$$

Here we assume $H_n \neq 0$ and $I_n \neq 0$. In the case of one of H_n or I_n is zero, one can argue similarly, and in fact it is easier. So we leave it to the reader. Combining the exact sequences (4) and (6), one gets the exact sequence,

$$0 \longrightarrow H_n \longrightarrow E \longrightarrow G'_n \longrightarrow 0. \quad (7)$$

Note that $E \in \mathcal{A}_{(i,0)} \subset {}^0\text{Per}(X/Y_i)$. Thus Lemma 3.5 (2) implies $R^1 g_{i*} E = 0$. Therefore the long exact sequence associated to (7) implies $R^1 g_{i*} G'_n = 0$. Again by Lemma 3.5, we have $G'_n \in {}^0\text{Per}(X/Y_i)$. This means the exact sequence (7) is also an exact sequence in $\mathcal{A}_{(i,0)}$. By the stability of E , one gets,

$$\phi(H_n) < \phi(E). \quad (8)$$

Then we apply Lemma 6.4 below. Note that because E is σ -stable, we have $\text{Hom}(E, E) = \mathbb{C}$. Therefore Lemma 6.3 implies $E \in \mathcal{S}$, and because H_n is a subsheaf of E , we have $H_n \in \mathcal{S}$. Furthermore, since E and H_n are objects of $\mathcal{A}_{(i,0)}$, one has

$$Z_{(\beta,\omega)}(E) \neq 0, \quad Z_{(\beta,\omega)}(H_n) \neq 0.$$

Let us take $\varepsilon' = \frac{1}{3}\varepsilon$. Then by Lemma 6.4, we can find $M' > 0$ which does not depend on E , such that $n > M'$ implies

$$|\phi_n(E) - \phi(E)| < \varepsilon', \quad |\phi_n(H_n) - \phi(H_n)| < \varepsilon'. \quad (9)$$

Because $I_n[1] \in {}^0\text{Per}(\mathcal{D}_{X/Y_i})$, we have

$$Z_{(\beta,\omega)}(I_n[1]) \in \mathbb{R} < 0.$$

Thus we apply Lemma 6.4 to get $0 < \phi_n(I_n) < \varepsilon'$ for $n > M'$. Combined with the sequence (6), we have either

$$0 < \phi_n(F_n) < \varepsilon' \quad \text{or} \quad (10)$$

$$\phi_n(H_n) \geq \phi_n(F_n). \quad (11)$$

In the first case, we have

$$\begin{aligned} |\phi_n(F_n) - \phi(E)| &\leq (\phi_n(F_n) - \phi_n(E)) + |\phi_n(E) - \phi(E)| \\ &< 2\varepsilon' < \varepsilon. \end{aligned}$$

Here we have used (5), (9) and (10). In the second case, we have

$$\begin{aligned} |\phi_n(F_n) - \phi(E)| &\leq (\phi_n(F_n) - \phi_n(E)) + |\phi_n(E) - \phi(E)| \\ &< (\phi_n(H_n) - \phi_n(E)) + \varepsilon' \quad \text{from (11), (9)} \\ &< (\phi(H_n) - \phi(E)) + 2\varepsilon' + \varepsilon' \quad \text{from (9)} \\ &< 3\varepsilon' = \varepsilon \quad \text{from (8)}. \end{aligned}$$

By the same argument, one can show $|\phi_n(F'_n) - \phi(E)| < \varepsilon$ for the semistable factor F'_n in σ' with the smallest phase, when $n > M'$. Therefore the condition (3) holds. \square

Here we have used the following lemma:

Lemma 6.4 *We fix $\beta + i\omega \in \mathcal{W}_i$ as in Proposition 4.4. Then for each $\varepsilon' > 0$, there exists $M' > 0$ such that $n > M'$ implies*

$$\sup \left\{ \left| \frac{Z_{(\beta, \omega_n)}(F)}{Z_{(\beta, \omega)}(F)} - 1 \right| : F \in \mathcal{S}, Z_{(\beta, \omega)}(F) \neq 0 \right\} < \varepsilon'.$$

Proof. One can calculate as follows:

$$\left| \frac{Z_{(\beta, \omega_n)}(F)}{Z_{(\beta, \omega)}(F)} - 1 \right| = \frac{1}{n} \cdot \frac{\text{ch}_2 F \cdot D_i}{|-\text{ch}_3 F + \beta \text{ch}_2 F + i\omega \text{ch}_2 F|}. \quad (12)$$

If $Z_{(\beta, \omega)}(F) \neq 0$, then $\omega \cdot \text{ch}_2(F) \neq 0$ or $-\text{ch}_3 F + \beta \text{ch}_2 F \neq 0$. In the former case, one has

$$(12) \leq \frac{1}{n} \cdot \frac{D_i \cdot \text{ch}_2 F}{\omega \cdot \text{ch}_2 F},$$

and there are only finite number of possibilities for the values

$$\left\{ \frac{D_i \cdot \text{ch}_2 F}{\omega \cdot \text{ch}_2 F} : F \in \mathcal{S}, \text{ch}_2 F \cdot \omega \neq 0 \right\}.$$

In the latter case, one has

$$(12) \leq \frac{1}{n} \cdot \frac{D_i \cdot \text{ch}_2 F}{|-\text{ch}_3 F + \beta \text{ch}_2 F|},$$

and because there are finite number of possibilities for the values $\{\beta \cdot \text{ch}_2 F : F \in \mathcal{S}\}$, we have

$$\inf\{|-\text{ch}_3 F + \beta \text{ch}_2 F| : F \in \mathcal{S}, -\text{ch}_3 F + \beta \text{ch}_2 F \neq 0\} > 0.$$

Therefore there exists a constant K' independent of $F \in \mathcal{S}$ with $Z_{(\beta, \omega)}(F) \neq 0$ such that $(12) \leq K'/n$. Hence for each ε' , one can take M' to be K'/ε . \square

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